Multivariate Option Pricing Using Copulae

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Abstract: The complexity of financial products significantly increased in the past ten years. In this paper we investigate the pricing of basket options and more generally of complex exotic contracts depending on multiple indices. Our approach assumes that the underlying assets evolve as dependent GARCH(1,1) processes. The dependence among the assets is modeled using a copula based on pair-copula constructions. Unlike most previous studies on this topic, we do not assume that the dependence observed between historical asset prices is similar to the dependence under the risk-neutral probability. The method is illustrated with US market data on basket options written on two or three international indices.

Keywords: Pair-copula construction, basket options, multivariate derivatives, pricing.

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1 Introduction

There has been significant innovation in the financial industry in the past ten years. More and more basket options and complex exotic contracts depending on multiple indices are issued. This paper proposes to evaluate multivariate derivatives. Our approach assumes that each underlying asset evolves as a GARCH(1,1) process. Unlike most previous studies on this topic, we do not assume that the dependence observed between historical prices is necessarily similar to the dependence under the risk-neutral probability. The method is implemented with market data from the New York Stock Exchange on basket options written on three international indices.

There exist several approaches to price options written on dependent assets. The first approach is to consider a multivariate Black-Scholes model. The setting consists of n assets modeled by multivariate geometric Brownian motions with constant volatility and constant interest rates. Another approach was proposed by Galichon [2006] who extends the idea of the local volatility model developed by Dupire [1994] to build a stochastic correlation model (see also Langnau [2009]). Rosenberg [2003] and Cherubini and Luciano [2002] propose a non-parametric estimation of the marginal risk-neutral densities (using option prices written on each asset). Van den Goorbergh et al. [2005] adopt a parametric approach. They estimate a GARCH(1,1) for each asset under the physical measure, and then use the transformation by Duan [1995] to obtain the risk-neutral distribution. Cherubini and Luciano [2002] and Van den Goorbergh et al. [2005] model the dependence between the different underlying assets using historical data on the joint distribution. The same dependence is then assumed under the risk-neutral probability. Cherubini and Luciano [2002] study digital binary options and Van den Goorbergh et al. [2005] apply their techniques on some hypothetical contracts written on the maximum or the minimum of two assets. Both papers study financial derivatives written on only two indices and have no empirical examples.

In this paper, we extend the paper of Van den Goorbergh et al. [2005] in several directions. First, we consider contracts with possibly more than two underlying indices and evaluate then using a pair-copula construction (Aas et al. [2009]). Second, we examine a dataset of basket options prices and investigate whether the multivariate copula of the underlying assets is the same under the objective measure P and under the risk-neutral measure Q. As far as we know all the previous studies using copulae make this assumption, except Galichon [2006] and Langnau [2009]. The latter authors model the dynamics of the assets directly under the risk-neutral probability, and their model fits perfectly the market prices by construction. Finally, we study the sensitivity of basket option prices to the choice of the parameters for the GARCH(1,1) processes, the copula family and of its parameters to understand the impact of dependence misspecification.

There are arguments to believe that the copula under the objective measure P is similar to the copula under the risk-neutral measure Q. For example, in the multivariate Black-Scholes model, the change of measure between P and Q does not influence the dependence. The covariance matrix stays the same, only the drift terms change. Rosenberg [2003] and Cherubini and Luciano [2002] argue that the dependence structure under Q will be the same as under P when the risk-neutral returns are increasing functions of the objective returns. Galichon [2006] argues in a very different way. To him, "it is an extreme assumption to make only in the extreme hypothesis where the market does not provide any supplemental information on the dependence structure, which is usually not the case (the price of basket options, for instance, contains information on the market price of the dependence structure)". However in his study, he does not explore this direction. Our setting is different from Rosenberg [2003] and Cherubini and Luciano [2002] in that we model assets with GARCH(1,1) processes and make use of Duan [1995]'s transformation. The change of measure of Duan [1995] has a particular effect on the GARCH(1,1) process. After this change, not only the drift is changed but also the volatilities. This transformation is not monotonic which suggests that the dependence may be different in the historical world and the risk-neutral world.

In this paper, we price an option written on more than two assets in a dynamic-copula setting. Dependence problems with more than two assets are significantly more difficult. However using pair-copula constructions, the problem comes back to study the dependence between two variables at a time. We illustrate the study with a concrete example using data from the North American financial market. This set of data is used to show how to implement our techniques and to discuss how the dependence structure under the objective measure and the risk-neutral world may be different. Our conclusions are preliminary as the dataset is limited. Section 2 presents the pricing of a bivariate option. Section 3 extends the study to an option written on more than two indices. Section 4 illustrates the techniques presented in the paper using quotes from the financial market for trivariate options. Finally, Section 5 shows that prices of multivariate options are very sensitive to the dependence structure and that a pair-copula construction can capture sensitivities that a standard trivariate Gaussian copula is not able to. We further illustrate that modelling dependence appropriately is not only important

when pricing multivariate derivatives but also when hedging them.

2 Bivariate Option Pricing

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In this section, we first recall how to price a European option that depends on the final value at maturity of two assets using a similar approach as Van den Goorbergh et al. [2005].

2.1 Distribution of the underlying assets under P

Denote by $S_i(t)$ the closing price of index *i* for the trading day *t*, and define the log-return on asset *i* for the tth trading day as

$$r_{i,t+1} = \log\left(S_i(t+1)/S_i(t)\right) \tag{1}$$

where i = 1 or i = 2. Let $\mathcal{F}_t = \sigma(r_{1,s}, r_{2,s}, s \leq t)$ denote all returns information available at time t. Similar to Van den Goorbergh et al. [2005], we assume that the marginal distributions of $S_1(.)$ and $S_2(.)$ respectively follow GARCH(1,1) processes with Gaussian innovations. The dependence structure between the standardized innovations up to time t is given by a copula $C_t^P(.,.)$ that may depend on time t and is defined under the physical probability measure P. This model is quite general and allows for time-varying dependence as well as time-varying volatilities in a non-deterministic way. Indeed the dependence can change with the volatility in the financial market (see Van den Goorbergh et al. [2005] for an example).

Under the objective measure P, the log-returns of each asset S_i for i = 1and i = 2 evolve as follows:

$$\begin{cases} r_{i,t+1} = \mu_i + \eta_{i,t+1}, \\ \sigma_{i,t+1}^2 = w_i + \beta_i \sigma_{i,t}^2 + \alpha_i (r_{i,t+1} - \mu_i)^2, \\ \eta_{i,t+1} | \mathcal{F}_t \sim_P N(0, \sigma_{i,t}^2) \end{cases}$$
(2)

where $w_i > 0$, $\alpha_i > 0$, $\beta_i > 0$ and $\alpha_i + \beta_i < 1$, and where \sim_P refers to the distribution under P. μ_i is the expected daily log-return for S_i . The GARCH parameters for each margin are estimated by maximum likelihood separately, using the unconditional variance level $\frac{w_i}{1-\beta_i-\alpha_i}$ as starting value $\sigma_{i,0}^2$. Denote the standardized innovations by

$$(Z_{1,s}, Z_{2,s})_{s \leqslant t} := \left(\frac{\eta_{1,s}}{\sigma_{1,s}}, \frac{\eta_{2,s}}{\sigma_{2,s}}\right)$$

The standardized innovations $(Z_{1,s})_s$ and $(Z_{2,s})_s$ are respectively i.i.d. with a standard normal distribution N(0, 1), but in general the process Z_1 is not independent of the process Z_2 . Let F_1^P be the cdf of $Z_{1,s}$ and F_2^P be the cdf of $Z_{2,s}$ under the objective measure P. Note that F_1^P and F_2^P are N(0, 1)-distributed in our specific case. In general, using Sklar [1959]'s theorem, the joint distribution F^P of Z_1 and Z_2 can be written as a function of its marginals. Precisely there exists a unique copula C^P , such that

$$F^{P}(z_{1}, z_{2}) = C^{P}(F_{1}^{P}(z_{1}), F_{2}^{P}(z_{2}))$$
(3)

for all $z_i \in \mathbb{R}$, i = 1, 2. We then assume that the copula $C^P(., .)$ is a parametric copula and θ_P corresponds to the parameter(s) of this copula. We propose to look at a wide class of parametric copulae, the Gaussian, *T*-Student, Clayton, Gumbel, Frank, Joe, BB1, BB6, BB7 and BB8 copulae as well as their respective rotated versions (by 90°, 180° or 270°) (see Joe [1997] and Brechmann and Schepsmeier [2011]) but it is straightforward to extend our study to other families of copulae.

2.2 Pricing of a bivariate option

Assume the financial market is arbitrage-free and denote by Q the chosen risk-neutral probability to perform the pricing of derivatives. Consider an option whose payoff depends only on the terminal values of two indices S_1 and S_2 . Let us denote by $g(S_1(T), S_2(T))$ its payoff. The price at time t of this derivative is given by

$$p_t = e^{-r_f(T-t)} E_Q \left[g(S_1(T), S_2(T)) | \mathcal{F}_t \right]$$
(4)

where E_Q denotes the expectation taken under the risk-neutral probability Q. Here r_f denotes the constant daily risk-free rate and T - t corresponds to the time to maturity calculated in number of trading days. The price (4) can also be expressed as a double integral

$$p_t = e^{-r_f(T-t)} \int_0^{+\infty} \int_0^{+\infty} g(s_1, s_2) f^Q(s_1, s_2) ds_1 ds_2$$

where f^Q denotes the joint density of $S_1(T)$ and $S_2(T)$ under the risk-neutral probability Q. Similar to (3), it is possible to express the joint density using the marginal densities f_1 and f_2 of respectively $S_1(T)$ and $S_2(T)$ as follows:

$$f^{Q}(x_{1}, x_{2}) = c^{Q}_{12}(F^{Q}_{1}(x_{1}), F^{Q}_{2}(x_{2}))f^{Q}_{1}(x_{1})f^{Q}_{2}(x_{2}).$$

Here $c_{12}^Q = \frac{\partial^2 C^Q(y_1, y_2)}{\partial y_1 \partial y_2}$ where $C^Q(., .)$ is the copula between $S_1(T)$ and $S_2(T)$ under Q. To value the option and calculate its price (4), one needs the *joint* distribution of $S_1(T)$ and $S_2(T)$ under Q, that is their respective marginal distributions F_1^Q and F_2^Q , as well as the copula C^Q .

Following Duan [1995] and Van den Goorbergh et al. [2005], and assuming that the conditions needed for the change of measure of Duan [1995] are satisfied, the log-returns under the risk-neutral probability measure Q are given as follows

$$\begin{cases} r_{i,t+1} = r_f - \frac{1}{2}\sigma_{i,t}^2 + \eta_{i,t+1}^*, \\ \sigma_{i,t+1}^2 = w_i + \beta_i \sigma_{i,t}^2 + \alpha_i (r_{i,t+1} - \mu_i)^2, \\ \eta_{i,t+1}^* | \mathcal{F}_t \sim_Q \mathcal{N}(0, \sigma_{i,t}^2) \end{cases}$$
(5)

where r_f is the daily constant risk-free rate. We assume that this change of measure is valid ¹. Note that the daily risk-free rate r_f plays a critical role in the simulation of the process in the risk-neutral world, and therefore in the pricing of the security. We need to control for the influence of significant changes in the level of the risk-free rate over the last years (see discussion in Section 4).

Dependence Modelling

To model the dependence under Q, there are two possible approaches. The first approach consists of assuming that it is similar to the dependence under P. As far as we know, this has been a standard assumption in the literature, see Cherubini and Luciano [2002], Chiou and Tsay [2008], Rosenberg [2003], and Van den Goorbergh et al. [2005]. Our approach is quite different. We would like to infer from market prices of bivariate options the joint distribution of assets under Q, and therefore the copula under Q. We assume that the copula under Q belongs to the same family as the copula used under Pbut we do not impose that they have the same parameters. Our approach is therefore parametric.

Assume for example that the copula under P is a parametric copula with one parameter θ_P . We investigate if the same copula with a possibly different parameter θ_Q could better reflect market movements in options' prices. Suppose that we observe p_t^M the market price of the option at time t. For

¹To apply Duan [1995]'s change of measure, we restrict ourselves to Gaussian innovations. In addition we assume that the conditional distribution of each asset to the entire information \mathcal{F}_t at time t is similar to the conditional distribution to the information generated solely by this asset up to time t. Duan [1995] shows that, under certain conditions, the change of measure comes down to a change in the drift.

any parameter θ_Q we can also calculate a Monte Carlo estimate $\hat{p}_t^{mc}(\theta_Q)$ of this price using formula (4). We then solve for the parameter $\theta_Q(t)$ of the copula such that the estimated price $\hat{p}_t^{mc}(\theta_Q(t))$ is as close as possible to the market price p_t^M . We can then compare θ_P with θ_Q to see whether they are significantly different.

Extension to time-varying dependence

In practice the dependence changes over time, in particular with the level of volatility on the market. When the volatility is high, the dependence is usually higher. It is possible to extend our approach to the case when the parameters of the copula are time-varying, precisely are function of the volatility observed in the market. For example, Van den Goorbergh et al. [2005] assume that

$$\theta_P(t) = f\left(\gamma_0 + \gamma_1 \log\left(\max(\sigma_{1,t}, \sigma_{2,t})\right)\right) \tag{6}$$

where f is a given function. Then there are two additional parameters γ_0 and γ_1 to fit and a specific study is needed each time to determine the best relationship (6) to assume between the volatilities in the market at time tand the copula parameter. Other time-varying copula models might involve GARCH components as in Ausin and Lopes [2010] or stochastic volatility components as in Hafner and Manner [2008] or Almeida and Czado [2012]. While Ausin and Lopes [2010] and Almeida and Czado [2012] use a Bayesian approach for estimation, the approach taken by Hafner and Manner [2008] involves efficient importance sampling. For the ease of exposition, we restrict ourselves to the case when the dependence is not time-varying. Our model could easily be extended to time-varying copulae by adding more parameters to the model.

We now extend the idea developed in this section to trivariate options in Section 3, and illustrate the study in Sections 4 and 5 with examples of basket options written on three indices.

3 Multivariate option pricing when there are more than two indices

We first describe pair-copula construction in the case of three indices. It is then illustrated with an example of a trivariate option.

3.1 Multivariate Dependence Modeling

There are many different approaches to model multivariate dependence. In this paper we continue to follow a copula approach. While there are many bivariate copulae the choice for multivariate copulae tended to be limited, especially with regard to asymmetric tail dependence among pairs of variables. Joe [1996] gave a construction method for multivariate copulae in terms of bivariate copulae. The bivariate building blocks represent bivariate margins as well as bivariate conditional distributions. Graphical methods to identify the necessary building blocks were subsequently developed by Bedford and Cooke [2001, 2002]. Their full potential to model different dependence structures for different pairs of variables is recognized by Aas et al. [2009] and applied to financial return data. This construction approach is called the pair-copula construction method for multivariate copulae. For three dimensions the construction method is simple and proceeds as follows. Let $f(x_1, x_2, x_3)$ denote the joint density, which is decomposed for example by conditioning as

$$f(x_1, x_2, x_3) = f(x_3 | x_1, x_2) \times f_{2|1}(x_2 | x_1) \times f_1(x_1).$$
(7)

Now by Sklar's theorem we have $f(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2)$ and therefore

$$f_{2|1}(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2))f_2(x_2).$$

Similarly we have $f_{3|1}(x_3|x_1) = c_{13}(F_1(x_1), F_3(x_3))f_3(x_3)$. Finally, we use Sklar's theorem for the conditional bivariate density

$$f(x_2, x_3|x_1) = c_{23|1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))f_{2|1}(x_2|x_1)f_{3|1}(x_3|x_1)$$

and therefore

$$f(x_3|x_1, x_2) = c_{23|1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))f_{3|1}(x_3|x_1).$$

Putting these expressions into (7) it follows that

$$f(x_1, x_2, x_3) = c_{12}(F_1(x_1), F_2(x_2))c_{13}(F_1(x_1), F_3(x_3))$$
(8)

$$\times \quad c_{23|1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))f_1(x_1)f_2(x_2)f_3(x_3). \quad (9)$$

Denote by $u_i = F_i(x_i)$ for i = 1, i = 2 and i = 3. The corresponding copula density is therefore given by

$$c_{123}(u_1, u_2, u_3) = c_{12}(u_1, u_2) \cdot c_{13}(u_1, u_3) \cdot c_{23|1}(F_{2|1}(u_2|u_1), F_{3|1}(u_3|u_1))$$
(10)

The copula with density given by (10) is called a D-vine in three dimensions and involves only bivariate copulae. More general pair-copula constructions are contained in Aas et al. [2009] and a recent survey on such constructions is given by Czado [2010]. Further extensions and applications are also provided in Kurowicka and Joe [2011].

We now show how to apply this technique to the valuation of options linked to three market indices.

3.2 Modelling of the underlying (S_1, S_2, S_3)

In this section, we describe each step needed to simulate the underlying (S_1, S_2, S_3) under P and infer the dependence structure under P. This dependence structure will then be used in the option pricing in Section 3.3. The first step consists of fitting a GARCH(1,1) process on each marginal using historical data.

Step 1: Calibration of the GARCH(1,1) processes.

At time t (valuation date of the option, say 3^{rd} of November 2009), we calibrate a GARCH process using Δ past informations, corresponding to the Δ trading days prior to t. For each underlying asset S_i , i = 1, 2, 3, we find $\hat{\mu}_i$, \hat{w}_i , $\hat{\alpha}_i$ and $\hat{\beta}_i$ as well as the daily volatilities $\hat{\sigma}_{i,s}$ for each time $t - \Delta < s \leq t$. The Δ estimated standardized innovations are then obtained as

$$(Z_{1,s}, Z_{2,s}, Z_{3,s})_{s \in]t-\Delta, t]} := \left(\frac{\hat{\eta}_{1,s}}{\hat{\sigma}_{1,s}}, \frac{\hat{\eta}_{2,s}}{\hat{\sigma}_{2,s}}, \frac{\hat{\eta}_{3,s}}{\hat{\sigma}_{3,s}}\right)$$

 $(Z_{i,s})_s$ denotes the stochastic process Z_i as a function of s. In the GARCH(1,1) model used to calibrate the marginals, the standardized innovations are N (0, 1). We obtained the corresponding estimated standardized innovations in the interval (0, 1) by applying Φ , the cdf of the standard normal distribution N(0, 1). Let us denote by U the corresponding variables

$$(U_{1,s}, U_{2,s}, U_{3,s})_s := \left(\Phi\left(\frac{\hat{\eta}_{1,s}}{\hat{\sigma}_{1,s}}\right), \Phi\left(\frac{\hat{\eta}_{2,s}}{\hat{\sigma}_{2,s}}\right), \Phi\left(\frac{\hat{\eta}_{3,s}}{\hat{\sigma}_{3,s}}\right)\right)_s.$$
(11)

The dependence structure between $(Z_{1,s})_s$, $(Z_{2,s})_s$ and $(Z_{3,s})_s$ is the same as between $(U_{1,s})_s$, $(U_{2,s})_s$ and $(U_{3,s})_s$ because a copula is invariant by a change by an increasing function (see Joe [1997] for instance).

Step 2: Dependence under *P*.

At time t, we fit a copula on the joint distribution of $(U_{1,s}, U_{2,s}, U_{3,s})_{t-\Delta < s \leq t}$ as follows. The copula density $c^P(u_1, u_2, u_3)$ is equal to

$$c_{12}(u_1, u_2; \theta_{12})c_{13}(u_1, u_3; \theta_{13})c_{23|1}\left(F_{2|1;\theta_{12}}(u_2|u_1; \theta_{12}), F_{3|1;\theta_{13}}(u_3|u_1; \theta_{13}); \theta_{23|1}\right)$$
(12)

where c_{12} , c_{13} and $c_{23|1}$ are three parametric copula densities with respective parameters θ_{12} , θ_{13} and $\theta_{23|1}$, and where $F_{2|1;\theta_{12}}$ denotes the conditional cdf of U_2 given $U_1 = u_1$. The estimates of the parameters $\hat{\theta}_{12}$, $\hat{\theta}_{13}$ and $\hat{\theta}_{23|1}$ depend on the time t at which the estimation is done and also on the size of the time window Δ (here we have Δ daily observations between $]t - \Delta, t]$). Note that θ_{α} where $\alpha = 12, 13$ or 23|1 is a generic notation for the parameter(s) of the copula and may represent a vector of parameters if the parametric copula depends on more than one parameter. To simplify, we restrict ourselves to well-known classes of one or two-parameter copulae. In the example, we investigate the Gaussian, T-Student, Gumbel, Clayton, Joe, Frank, BB1, BB6, BB7, BB8 as well as their rotated versions (by 90°, 180° (survival) and 270°). To do so we use the R-package of Brechmann and Schepsmeier [2011].

Notice also that the decomposition (12) depends on the order of the variables. We discuss later methods to determine an appropriate order. For illustration we assume the order of the variables to be S_1 , S_2 and S_3 . The dependence between S_1 and S_2 is modeled by c_{12} and the one between S_1 and S_3 by c_{13} . We further examine $c_{23|1}$ which is the dependence between S_2 and S_3 conditional to S_1 . The conditional distribution cannot be observed directly. To obtain pseudo observations that are distributed along the conditional distribution, we use (11) and calculate for each observation $s \in [t - \Delta, t]$,

$$u_{2|1s} := F_{2|1;\theta_{12}}(u_{2s}|u_{1s};\theta_{12}^{P}) u_{3|1s} := F_{3|1;\theta_{13}}(u_{3s}|u_{1s};\widehat{\theta}_{13}^{P})$$
(13)

where the conditional distribution $F_{2|1;\hat{\theta}_{12}^P}$ is obtained by

$$F_{2|1;\widehat{\theta}_{12}^{P}}(u_{2}|u_{1};\widehat{\theta}_{12}^{P}) = \frac{\partial}{\partial u_{1}}C_{12}(u_{2}|u_{1};\widehat{\theta}_{12}^{P}) =: h_{u_{1}}(u_{2};\widehat{\theta}_{12}^{P})$$
(14)

and $F_{3|1;\hat{\theta}_{13}^P}$ similarly. We refer to Aas et al. [2009] where estimation and simulation algorithms for pair-copula constructions are provided and discussed. In particular the "h" functions needed in (14) are given in Aas et al. [2009] for the Gaussian, Clayton, Gumbel and *T*-Student distributions, in Czado et al. [2012] for the BB1 and BB7 copulae. They can easily be calculated for other copulae (using Joe [1996]). Step 2 is completed when the dependence structure is chosen and that the estimated parameters are calculated $(\hat{\theta}_{12}^P, \hat{\theta}_{13}^P \text{ and } \hat{\theta}_{23|1}^P)$. The superscript P recalls that the estimation of the dependence structure is obtained from historical data of the assets' prices, therefore it corresponds to the dependence structure under the objective measure P.

The pair-copula construction can easily be extended to higher dimensions. Bedford and Cooke [2001, 2002] introduced for this the notion of regular (R) vines using a sequence of linked trees for the identification of the bivariate building blocks. As the tree number increases the number of conditioning variables increases as well. In the first tree no conditioning is needed, while in the last tree the number of conditioning variables is d-2, where d is dimension of the data. Two simple subclasses of R-vines are often considered, called C-and D-vines. D-vines are especially useful for time ordered variables, while C-vines require the existence of a root node for each tree (see Czado [2010] for an easy derivation of their joint density).

We discuss now model selection choices of pair-copula constructions in three dimensions. First, in three dimensions the tree structure of C-, D- and R-vines coincide. Only the order of the variables, the family of pair-copulae and their parameters need to be chosen. As et al. [2009] suggested for Dvines to put the pairs with the strongest dependence in the first tree. This is done to have a parsimonious model and enhances further estimation stability. Dißmann et al. [2011] extended this idea to general *R*-vines in a tree wise fashion. It starts by giving all possible pairs a weight such as the absolute value of the empirical Kendall's. Then they applied a maximal spanning tree algorithm to determine a tree with a maximal sum of weights and set this tree to the first tree of the R-vine. Standard information criteria such as AIC and BIC are then used to select the best fitting pair-copula family from a set of considered pair-copulas for each edge pair in the first tree. Finally, the corresponding parameters are estimated using either inversion of Kendall's tau or maximum likelihood for each pair. Given the results of the first tree, a sample for the conditional bivariate distributions are created (compare to (13)) and the next tree is selected in a similar fashion among all pairs allowed by the proximity condition necessary for R-vines (see Bedford and Cooke (2002)).

Given the tree structure and the pair-copula families full maximum likelihood estimation of the parameters is feasible up to 20 dimensions, however it can be extended to even higher dimensions if one restricts the choice of paircopulae in higher trees (see Heinen and Valdesogo [2009], Brechmann et al. [2012] and Brechmann and Czado [2011]) and resorts to sequential estimation of the copula parameters (see Czado et al. [2012] for C-vines and Haff [2012] for D-vines). Dißmann et al. [2011] extended the sequential estimation of the copula parameters to general R-vines.

Steps 1 and 2 correspond to an inference for margins approach (IFM) introduced by Joe [2005] and Joe and Xu (1996) commonly used for the estimation of marginal and copula parameters. Here marginal parameters are estimated first and an approximative sample from a copula distributions (compare to (11)) is formed. Estimation of the copula parameters is then based on this sample. Kim et al. [2007] demonstrate that one needs a strong misspecification in the marginal model for the IFM approach to fail in the estimation of the copula parameters. If the *R*-vine model for the dependence part is fully specified and full maximum likelihood is used for the copula parameters based on copula data, then this is exact IFM. If we use sequential estimation for the copula parameters we extend the IFM method by using a different estimation method. The asymptotic behavior of this approach is studied for *D*-vines in Haff [2012].

3.3 Pricing of a Trivariate Option

In this section, we describe how to price an option written on three market indices. This methodology will then be applied to examples in Sections 4 and 5. To price an option one needs the dynamics of the underlying indices under the probability measure Q. We assume that the copula under Qbelongs to the same family as the one determined under P, but may have different parameters. In Step 3, we describe how to simulate the underlying indices (S_1, S_2, S_3) under Q for an arbitrary set of parameters θ_{12}^Q , θ_{13}^Q and $\theta_{23|1}^Q$. In Step 4, we explain how to simulate an option price and calibrate at the same time the model (and therefore determine θ_{12}^Q , θ_{13}^Q and $\theta_{23|1}^Q$).

Step 3: Simulation of (S_1, S_2, S_3) under Q.

Given a set of parameters θ_{12}^Q , θ_{13}^Q and $\theta_{23|1}^Q$, we simulate observations from the D-vine specification (10)

$$\left(U_{1,s}^Q, U_{2,s}^Q, U_{3,s}^Q\right)_{s < t \leqslant T} \tag{15}$$

with the dependence structure identified in Step 2, but with a parameter set

$$\Theta^Q := \left(\theta^Q_{12}, \theta^Q_{13}, \theta^Q_{23|1}\right). \tag{16}$$

To simulate from a D-vine, we refer to Algorithm 2 on page 187 of Aas et al. [2009]. From the D-vine data (15), we can obtain recursively the standardized

residuals $Z_{i,s}^Q$ such that

$$Z_{i,s}^{Q} = \Phi^{-1}\left(U_{i,s}^{Q}\right) = \frac{r_{i,s} - r_f + \frac{\sigma_{i,s-1}^{2}}{2}}{\sigma_{i,s}}.$$
(17)

This is a consequence of the dynamics (5) under Q. The algorithm is recursive. Given initial volatilities $\sigma_{i,0}$ for i = 1, 2, 3 (equal for instance to the square root of the unconditional variance level $\frac{w_i}{1-\beta_i-\alpha_i}$), one can get the first innovation $r_{i,1}$ using the first line of (5). Then using the second line of (5) for given α_i , β_i and ω_i , one computes $\sigma_{i,1}^2$ for i = 1, 2, 3. Then, $Z_{i,1}^Q$ is obtained by the ratio of $r_{i,1}$ and $\sigma_{i,1}$. The full process is then constructed recursively.

Step 4: Model Calibration and Pricing.

The option price at time t is equal to

$$p_t = e^{-r_f(T-t)} E_Q \left[g(S_1(T), S_2(T), S_3(T)) | \mathcal{F}_t \right].$$
(18)

For a given parameter Θ^Q in (16) we simulate $(S_1(T), S_2(T), S_3(T))$ from Step 3. To estimate the price at time t, p_t , one uses as inputs the prices $S_i(t)$ observed in the financial market at time t. An estimate $\hat{p}_t^{mc}(\Theta^Q)$ of (18) is then calculated by Monte Carlo methods.

The remaining question is about the choice of the parameter set Θ^Q . Estimating this parameter set corresponds to calibrating the pricing model. Let us relate this to a simpler and well-known problem. Assume that the pricing model is the Black-Scholes model, the pricing formula of an option written on a single stock depends on the risk-free rate r and the volatility σ . The volatility parameter σ such that the market price of the option is equal to the model price (Black-Scholes formula) is called the "implied volatility". In a multidimensional Black-Scholes model, one would have an "implied correlation matrix". In our setting we have the set of parameters Θ^Q that we can infer from market data, and therefore an "implied" copula under Q.

Recall that prices depend on time. Consider past observations of prices of the trivariate option, say at dates t_i , i = 1..n, the set of parameters Θ^Q needed to characterize the copula C^Q is calculated at time t such that it minimizes the sum of quadratic errors

$$\min_{\Theta^Q} \sum_{i=1}^{n} \left(\hat{p}_{t_i}^{mc}(\Theta^Q) - p_{t_i}^M \right)^2$$
(19)

where $p_{t_i}^M$ denotes the market price of the trivariate option at the date t_i , and $\hat{p}_{t_i}^{mc}$ is the Monte Carlo estimate of its price obtained by the procedure described in Step 3.

In the case of a path-dependent contract, the technique is similar, where the price at t given in (18) is now calculated as

$$e^{-r_f(T-t)} E_Q \left[g \left(\left\{ S_i(s)_{s \in [0,T]} \right\}_i \right) | \mathcal{F}_t \right].$$

4 Empirical Analysis

We first describe the data, then present our numerical results.

4.1 Description of the Data

Our data come from the secondary market for exchange-listed structured products on the New York Stock Exchange². In May 2008, there were 24 index-linked notes written on multiple market indices (for a total volume of US\$590 million). We selected two structured products to illustrate our study. Both products are linked to three indices and were quoted daily³.

We now describe these trivariate basket options for which we have daily quotes of the prices from their respective issuance date to November 2, 2009. Daily log-returns of each index involved in these structured products are also available over the period under study. *MIB* and *IIL* are "Capital Protected Notes Based on the Value of a Basket of Three Indices". *MIB* and *IIL* are two similar products issued by Morgan Stanley, therefore we only describe one of them. The notes *IIL* are linked to the *Dow Jones EURO STOXX* 50^{SM} Index, the *S&P500* Index, and the *Nikkei 225* Index (let us denote them respectively by S_1 , S_2 and S_3). They were issued on July 31st, 2006 at an initial price of \$10 and matured on July 20, 2010 (which correspond roughly to 1,006 trading days). Their final payoff is given by

$$\$10 + \$10 \max\left(\frac{m_1 S_1(T) + m_2 S_2(T) + m_3 S_3(T) - 10}{10}, 0\right)$$
(20)

 $^{^2 \}mathrm{See}$ Bernard et al. [2011] for more information about these exchange-listed structured products.

³All information about these products is contained in the official prospectus supplements that were publicly available on www.amex.com and now listed on www.nyse.com. They can also be obtained upon request from the authors.

where $m_i = \frac{10}{3S_i(0)}$ such that $m_1S_1(0) + m_2S_2(0) + m_3S_3(0) = 10$ and the percentage weighting in the basket is 33.33% for each index. On July 31st, 2006, $m_1 = 0.000917803$, $m_2 = 0.002643329$, and $m_3 = 0.000222122$.

4.2 Estimation of r_f

We now describe how to estimate the risk-free rate. Duan [1995] showed that a GARCH(1,1) model could reflect well the implied volatility surface (the smile with respect to the strike and the decay with respect to the time to maturity). Unsurprisingly it gives reasonable estimates of the implied risk-free rate to price structured products written on one index.

The contracts *MIB* and *IIL* were five-year contracts issued by Morgan Stanley. Both are five-year contracts. To check whether the GARCH(1,1) model gives reasonable prices, we consider other contracts issued by Morgan Stanley, with similar long-term maturity, but written on a single index. Our estimate of the risk-free rate is then not influenced by the modeling of the dependence. To do so, we use the contract *PDJ* written on the Dow Jones Industrial Average, *DJIA* (issued on Feb 25, 2004 with maturity date of Dec 30, 2011), and the contract *PEL* written on the *S&P500* (issued on March 25th, 2004 with maturity date of Dec 30, 2011). Both contracts pay semi-annual coupons of respectively 0.4% and 0.5% (at the end of June and end of December) and their final payoff is calculated as

$$\$10 + \$10 \max\left(0, \frac{\frac{1}{8}\sum_{i=1}^{8} S_{t_i} - S_0}{S_0}\right)$$

where S_0 is the initial value of the underlying at the issuing date, and where $t_i = 30^{th}$ December of each year (starting in 2004 and ending in 2011).

For each of these contracts, we fit a GARCH(1,1) process based on a time window Δ of 250 days (about one year of data) on respectively historical data of *S&P500* and *DJIA*. We then use Duan's [1995] change of measure given by (5) and simulate the price using different values for a continuously compounded risk-free rate $r \in (1\%, 10\%)$. We then solve for the value of r such that the model price coincides with the market price. We did this calculation for 5 dates for each contract, 31^{th} December 2004, 2005, 2006, 2007 and 2008.

In Figure 1, we represent simulated prices on Dec 31st, 2004 of the contract PEL with respect to the risk-free rate. The higher r_f the least valuable the contract is. It illustrates the importance of controlling the effect of the risk-free rate if we want to discuss the change in the dependence structure under P and under Q. Here the "implied risk-free rate" r is about 4.03% on December 31, 2004.



Figure 1: Price obtained by the model of Duan (1995) for the contract PEL as a function of the risk-free rate r (with 10,000 Monte Carlo simulations to draw this graph). On Dec 31st, 2004, it is quoted at \$9.60.

Date	12/31/04	12/31/05	12/31/06	12/31/07	12/31/08
r for PDJ	4.3%	5.0%	6.1%	4.1%	2.0%
r for PEL	4.0%	4.6%	5.3%	3.4%	2.6%
ZC yield	4.05%	4.82%	5.03%	3.85%	2.3%

Table 1: Implied risk-free rate used to price the contracts PDJ and PEL, expressed as a continuously compounded annual rate. We also report the continuously compounded rates of the US zero-coupon yield curve (the daily rate is obtained by $r_f = r/250$). Note that it is not exactly a "daily" rate but the time step is a "trading day".

In Table 1 we report the values for the "implied risk-free rate" r such that the respective market values of the contracts are approximately equal to the estimates obtained by Monte Carlo simulations. We also report the corresponding rate from the US yield curve. For example, at the end of December 2004, the time to maturity of the *PEL* contract is about 1754

days (with an approximation of 250 trading days per year for the period after November 2009). A zero-coupon bond (from the US yield curve) of such maturity yields a continuously compounded interest rate of 4.05% per annum. Note that the differences between the US yield curve and the implied risk-free rates for *PDJ* and *PEL* are small. In addition these contracts are subject to default risk, therefore there could be a risk premium embedded in the interest rate that we neglect here.

This preliminary study shows that the GARCH(1,1) model is able to reproduce accurately the market prices and that the risk-free rate used in the pricing could either be obtained as an implied interest rate or directly from the US yield curve. Note that prices at issue are hard to reproduce and to fit because they include commissions. On purpose, we choose to evaluate the contracts at several dates posterior to the issuance date by several months.

4.3 Contract IIL

We now apply the four steps that we described in Section 3 to study the contract *IIL* in details.

Step 1: Calibration of the GARCH(1,1) processes.

We first fit a GARCH(1,1) over the entire period from July 2006 to November 2009 on the daily log-returns of S_1 (*Dow Jones EURO STOXX* 50^{SM}), S_2 (*S&P500* index), and S_3 (*NIKKEI 225* index). We then split the period into 3 subperiods of 290 trading days(from March 2006 to April 2007, from May 2007 to July 2008, and from August 2008 to November 2009) and fit a GARCH(1,1) for each subperiod. In Table 2, the estimates of the GARCH(1,1) are reported. The first column corresponds to the full period of observations (March 2006 to November 2009). The three other columns correspond to the three subperiods previously described. We note that in all cases, $\alpha_i + \beta_i$ is close to 1 but strictly less than 1, as it should be. Note also that in the second subperiod, the daily log-returns are on average negative or very close to 0, because it includes the recent financial crisis.

In addition to the parameters of the GARCH process, we also give the average daily volatilities for the full sample and for each subperiods. It is calculated as

$$\bar{\sigma}_{i,t} := \frac{1}{K} \sum_{s=t}^{t+K} \hat{\sigma}_{i,s}$$

multiplied by $\sqrt{250}$ to obtain an annual volatility (rather than a daily volatility), and where K denotes the number of days in the observation period. It

	Full sample	period 1	period 2	period 3
$\hat{\mu}_1$	0.000350	0.000907	-0.000494	0.000743
$\hat{\omega}_1$	3.76e-06	9.88e-06	1.027e-05	7.57e-06
$\hat{\alpha}_1$	0.1343	0.1598	0.1482	0.1062
\hat{eta}_1	0.8575	0.7275	0.8063	0.8854
$\hat{\mu}_2$	0.000414	0.000664	-0.000513	0.000593
$\hat{\omega}_2$	1.85e-06	2.72e-06	8.95e-06	5.42e-06
$\hat{\alpha}_2$	0.0932	0.0338	0.0513	0.119
$\hat{\beta}_2$	0.900	0.903	0.899	0.876
$\hat{\mu}_3$	0.000107	0.000525	-0.000594	0.0000213
$\hat{\omega}_3$	4.63e-06	4.75e-06	6.09e-06	1.83e-05
\hat{lpha}_3	0.127	0.0643	0.142	0.197
\hat{eta}_3	0.863	0.896	0.851	0.782
$\bar{\sigma}_{1,t}\sqrt{250}$	24.8%	14.5%	21.2%	38.7%
$\bar{\sigma}_{2,t}\sqrt{250}$	23.2%	10.3%	20.6%	38.8%
$\bar{\sigma}_{3,t}\sqrt{250}$	27.4%	17.5%	25.8%	39.1%

is striking how the volatility changes throughout the three periods. The dependence is also changing drastically such as it is described in Step 2.

Table 2: Estimated parameters of GARCH(1,1) for S_1 (*STOXX50*), S_2 (*S&P500*) and S_3 (*NIK225*). $\bar{\sigma}_{i,t}$ denotes the average of the daily volatilities over the period under study (July 2006 to November 2009).

Step 2: Dependence under *P*.

We first investigated the strength of the dependence between each pair out of $\{S_1, S_2, S_3\}$. The strongest dependence appears to be between S_1 (European index) and S_2 (US Index), then between S_1 (European index) and S_3 (Asian index). This can be seen from the values of Kendall's tau reported in Table 3. Table 3 also shows that Kendall's tau depends on the period.

To further explore the dependence structure including tail behavior, we constructed bivariate scatter plots of the copula data $(U_{1,s}, U_{2,s}, U_{3,s})$ of (11) for the three subperiods together with empirical contour plots for pairs of standardized innovations $(Z_{1,s}, Z_{2,s}, Z_{3,s})$. From the non elliptical shapes of some contour plots in Figure 2 we see that non-symmetry and tail dependence are visible, thus a more general D-vine model for the dependence will be more appropriate than a Gaussian model.

Recall that D-,C- and R-vines coincide in three dimensions and thus we

	$S_1 - S_2$	$S_1 - S_3$	$S_2 - S_3$
Full	0.404	0.202	0.079
Period 1	0.314	0.197	0.104
Period 2	0.384	0.239	0.075
Period 3	0.495	0.181	0.062

Table 3: Overall dependence measured by Kendall's tau for the full sample and then for each of the 3 periods.

use the general R-vine selection procedure of Dißmann et al. [2011] to select an appropriate D-vine to the copula data at each subperiod. In particular the selection procedure in three dimensions is sequential, i.e the first two pairs of variables are selected, which have the largest absolute dependence and then the resulting conditional pair-copula. For the data at hand this will be according to Table 3 the pairs (1, 2) and (1, 3) for the unconditional paircopulae, while the resulting conditional pair-copula is $c_{23|1}$. Next we have to choose the pair-copula families for c_{12} and c_{13} , respectively. This will be done using the Akaike information criterion (AIC), for which Brechmann [2010] has shown the best performance among several alternatives. Allowable bivariate copula families were the independence, Gauss, T-Student, Clayton, Gumbel, Frank, Joe, BB1, BB6, BB7, BB8, survival Clayton and survival Gumbel copula. Corresponding copula parameters are estimated by maximizing the bivariate likelihood. Now only the copula family choice for $c_{23|1}$ remains; for this pseudo observations as defined in (13) are formed based on the chosen pair copula families and their parameters estimates for c_{12} and c_{13} . Finally, the copula family is chosen again by AIC and the corresponding parameters are estimated by maximizing the bivariate likelihood. Since the marginal parameters are not reestimated when the copula parameters are estimated this correspond to a IFM type approach as already discussed earlier.

Figure 2: Scatterplot and normalized empirical contour plots of pairs of standardized innovations for each pair. Panel A,B,C correspond to 3 subperiods respectively from March 2006 to May 2007, from May 2007 to July 2008, and from August 2008 to November 2009.



Panel C: Third Period



The results of the above selection procedure for each subperiod are given in Table 4. As expected we see evidence of asymmetric tail dependence especially for the unconditional pair-copulae. The fit for the unconditional pair-copulae is good, when compared to empirical Kendall's τ values.

Period 1	c_{12}	c_{13}	$C_{23 1}$
Family	BB1	Gauss	Frank
$\hat{ heta}$	(0.14, 1.48)	0.30	0.08
$ au(\hat{ heta})$.35	.20	.01
empirical τ	.31	.20	-
Period 2	C ₁₂	C_{13}	$c_{23 1}$
Family	T-Student	SGumbel	Frank
$\hat{ heta}$	(0.57, 11.3)	1.26	-1.06
$ au(\hat{ heta})$.39	.21	12
empirical τ	.38	.24	-
Period 3	C ₁₂	c_{13}	$C_{23 1}$
Family	Gauss	BB7	Frank
$\hat{ heta}$.70	(1.11, .32)	-1.06
$ au(\hat{ heta})$.50	.18	12
empirical τ	.50	.18	-

Table 4: Selected vine models with pair-copula choice, parameter estimates and corresponding estimated (un)conditional Kendall's τ . For comparison empirical Kendall's τ s of unconditional pairs in the vine specification are also provided.

Step 3 and 4: Pricing the basket option

For the purpose of illustration, the pricing of the contract *IIL* is done at the end of the first period and we use the GARCH(1,1) fitted on the 1st period in Table 2. We perform also the pricing at the end of the second period, on the 1st of August 2008 and use the GARCH(1,1) parameters fitted on the 2nd period in Table 2. Our results are reported in Table 5. They are obtained using Monte Carlo simulations with 10,000 simulated trajectories for each underlying asset.

It is clear that the Monte Carlo estimates reported in Table 5 do not match the market prices. One explanation may come from the fact that the dependence under Q is indeed different from the dependence under P. One could change the parameters of the copulae appearing in Table 4 so that the Monte Carlo estimates match the market quotes. However there are many other potential explanations for the discrepancy observed in Table 5.

Another explanation may come indeed from the presence of credit risk. The contract guarantees a minimum payoff of 10 and an additional option payoff. For example a quote of 9.93 on August 1st, 2008 reflects an annual interest rate r higher than 3.5% (calculated as $9.93 = 10e^{-495r/252}$). To calculate this number we observe that there were about 495 trading days left before the maturity of the product and an option has a non-negative value. This is not possible unless the risk-free rate reflects the high credit risk at that time of the contract's issuer.

Moreover the contracts *MIB* and *IIL* are retail investment products. The secondary market for these markets has been criticized for not being liquid. In this market, issuers "choose" market prices. This may explain why these contracts appear underpriced.

	Nov. 4th, 2007	Aug. 1st, 2008	
	(end period 1)	(end period 2)	
Market price	11.16	0.02	
(quote from the NYSE)	11.10	9.90	
Monte Carlo Estimate	11.8 (0.05)	10.4.(0.05)	
using parameters in Tables 2 and 4	11.0 (0.05)	10.4(0.00)	
Monte Carlo Estimate with a	11.3 (0.06)	10.3 (0.06)	
Multivariate Gaussian Distribution	11.5 (0.00)	10.5 (0.00)	

Table 5: Pricing of IIL contract on November 4th, 2007 and on August 1st, 2008. Standard deviations for the Monte Carlo estimates are reported in parenthesis.

The last line of Table 5 corresponds to the Gaussian multivariate case. To perform the comparison, we need to describe a multivariate Gaussian distribution in the context of a pair-copula construction. Assume that the innovations $(Z_{1,s}, Z_{2,s}, Z_{3,s})$ is a multivariate normal distribution with mean (0, 0, 0) and covariance matrix

$$\begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}$$
(21)

One has $\tau_{12} = \frac{2}{\pi} \arcsin(\rho_{12})$ and $\tau_{13} = \frac{2}{\pi} \arcsin(\rho_{13})$. The conditional distribution of (X_2, X_3) given $X_1 = x$ is a bivariate normal distribution with mean $\begin{bmatrix} x\rho_{12} \\ x\rho_{13} \end{bmatrix}$ and variance $\begin{bmatrix} 1-\rho_{12}^2 & \rho_{23}-\rho_{12}\rho_{13} \\ \rho_{23}-\rho_{12}\rho_{13} & 1-\rho_{13}^2 \end{bmatrix}$ so that $\tau_{23|1} =$

 $\frac{2}{\pi} \arcsin\left(\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{13}^2}}\right).$ Then we can easily get ρ_{23} as $\rho_{23} = \rho_{12}\rho_{13} + \sin\left(\frac{\pi\tau_{23|1}}{2}\right)\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{13}^2}.$

The paper presents a methodology to price multivariate derivatives with dependent GARCH(1,1) processes. However to draw firm conclusions on whether the dependence under P is the same as the dependence under Q, more data (and better data) is needed. The next section illustrates how the price of a multivariate derivative is strongly affected by the dependence structure and therefore why it is important to not only model dependence using the Gaussian or the T-Student copula.

5 Additional Examples

In this paper we have used pair-copula constructions to model and fit the dependence between more than two indices. The use of a pair-copula construction accommodates a much wider range of dependence structure than the multivariate Gaussian copula. To illustrate this last point we present two additional examples of trivariate derivatives written on 3 indices S_1 , S_2 and S_3 . In particular these examples show that a misspecification of the dependence can have an important effect on the price of the derivative.

5.1 Comparison of Copulae

We assume S_1 , S_2 and S_3 are individually modeled by GARCH(1,1) processes with Gaussian innovations so that we can apply Duan's [1995] change of measure to perform risk-neutral pricing. Denote by τ_{12} and τ_{13} the Kendall's tau between the standardized innovations of S_1 and S_2 , respectively of S_1 and S_3 . Denote by $\tau_{23|1}$ the Kendall's tau for the distribution of the innovations of S_2 and S_3 conditional on the innovations of S_1 .

Table 6 presents 6 different assumptions on the dependence structure. In Scenario 1 of Table 6, we assume that the innovations follow a multivariate normal distribution since it is often used by practitioners. The correspondence between the usual parameters of a multivariate Gaussian distribution and the pair-copula construction is described in the previous section following (21). We then investigate the case when the dependence is based on the T-Student copula with a small degree of freedom (say 3). Finally, we look at more general pair-copula constructions. For the dependence structure, we compare the copulae reported in Table 6. To do so, we choose to vary τ_{12} while keeping τ_{13} and $\tau_{23|1}$ constant. For the GARCH(1,1) parameters, we choose for μ_i , ω_i , α_i and β_i the parameters fitted in Period 3 in Table 2 because they correspond to the most volatile period and dependence is more critical when the volatility in the market is high. Furthermore assume that $S_1(0) = S_2(0) = S_3(0) = 100$.

Scenario 1 corresponds to the multivariate Gaussian distribution.			
Scenarios	$S_1 - S_2$	$S_1 - S_3$	$S_2, S_3 S_1$
1	Gauss	Gauss, $\tau_{12} > 0$ $\tau_{13} = 0.7$	Gauss
2	T -Student $df = 3$ $\rho_{12} > 0$	<i>T</i> -Student	T-Student
		$df = 3, \tau_{13} = 0.7$	$df = 3, \tau_{23 1}$
3	Clayton for $\tau_{1,2} > 0$	$ au_{13} = 0.7$	
4	Gumbel for $\tau_{1,2} > 0$	Gumbel	Gumbel
	,	$\frac{\tau_{13} = 0.7}{\text{SClavton}}$	$\tau_{23 1}$ SClavton
5	SClayton for $\tau_{1,2} > 0$	$ au_{13} = 0.7$	$ au_{23 1}$
6	SGumbel for $\tau_{1,2} > 0$	SGumbel	SGumbel
		$ au_{13} = 0.7$	

Table 6: Copulae compared in Figures 3 and 4. "SGumbel" stands for the survival Gumbel copula (respectively "SClayton" refers to the survival Clayton copula). Scenario 1 corresponds to the multivariate Gaussian distribution.

Pair-Copula Constructions

In scenario 2 of Table 6, we assume that the dependence between Z_1 and Z_2 , between Z_2 and Z_3 and between Z_2 and Z_3 conditional on Z_1 are all T-Student copulae with d = 3 degrees of freedom. In this scenario one has $\tau_{12} = \frac{2}{\pi} \arcsin(\rho_{12}), \tau_{13} = \frac{2}{\pi} \arcsin(\rho_{13})$ and $\tau_{23|1} = \frac{2}{\pi} \arcsin(\rho_{23|1})$. In scenarios 3, 4, 5 and 6 we investigate the Clayton and survival Clayton, as well as the Gumbel and the survival Gumbel. They are one-parameter copulae and there is a bijection between the parameter of the copula and Kendall's tau (see for example Brechmann and Schepsmeier (2011)).

5.2 Multivariate Options Pricing

In this section we consider several multivariate contracts. Consider two derivatives linked to the maximum of 3 assets. Let $X_1(T)$ and $X_2(T)$ denote their respective payoffs paid at maturity T

$$X_1(T) = \max\left(\max\{S_1(T), S_2(T), S_3(T)\} - 100, 0\right)$$
(22)

or

$$X_2(T) = \max\{S_1(T), S_2(T), S_3(T)\} - \min\{S_1(T), S_2(T), S_3(T)\}$$
(23)

It is clear from Figures 3 and 4 that the use of the Gaussian copula or the T-Student copula tend to underestimate the price of the derivatives (22) and (23) written on the maximum when the market does not follow a multivariate Gaussian distribution. Note that for these derivatives the Clayton dependence tend to give much higher prices than the Gaussian dependence. This is consistent with the findings of Van den Goorbergh et al. (2005) for bivariate derivatives. If the market follows a multivariate Gaussian dependence, then the Clayton copula would overestimate the prices of the multivariate derivatives X_1 and X_2 studied in this section.

Insert here Figures 3 and 4 with prices for X_1 and X_2 .

For a given Kendall's tau in the market (or for a given Pearson correlation), the different scenarios give different prices. Using pair-copula construction may therefore explain prices in the market that are higher than the ones obtained with the Gaussian dependence structure. It is therefore of utmost importance to use a flexible and accurate dependence structure for that type of options.

5.3 Multivariate Options Hedging

Practitioners are not only interested in pricing derivatives but also in hedging them. Hedging is a very important issue. Indeed it is not enough for banks to sell derivatives at the "right" price, then they need to hedge the payoff of these derivatives using the premium they receive at time 0.

Here, we propose to delta-hedge the derivatives X_1 and X_2 . Delta-hedging is a dynamic strategy. Theoretically it has to be implemented continuously. In practice, one decides of a rebalancing frequency for the hedging portfolio. At each rebalancing date, it requires to compute the delta Δ_t of the option (or sensitivity of the option price to the underlying price). This consists of differentiating with respect to $S_i(t)$ the price p_t of the derivative at time 0 calculated earlier in (4). For example at time 0, we compute

$$\Delta_{i,t} := \frac{\partial p_t(S_1(t), S_2(t), S_3(t))}{\partial S_i(t)}$$

where $p_t(S_1(t), S_2(t), S_3(t))$ gives the price at time t as a function of the underlying prices $S_1(t), S_2(t)$ and $S_3(t)$. Practically $\Delta_{i,t}$ is approximated by finite difference. For example $\Delta_{1,0}$ is approximated by

$$\frac{p_0(S_1(0) + \varepsilon, S_2(0), S_3(0)) - p_0(S_1(0), S_2(0), S_3(0))}{\varepsilon}$$

for a small value of $\varepsilon > 0$. The efficiency of the delta-hedging strategy is related to the accuracy in the estimation for $\Delta_{i,t}$. A mistake in the estimation of the hedge ratio will therefore give rise to errors in the hedging and potential losses for the seller of the option. The following example stresses the importance of using the right dependence structure not only for the pricing but also in order to get an appropriate hedge. In Figure 5 and Figure 6 we report Monte Carlo estimates of $\Delta_{1,0}$ for the payoffs X_1 and X_2 under the different assumptions on dependence listed in Table 6.

Insert here Figure 5 and Figure 6 with hedge ratios for X_1 and X_2 .

From the graphs in Figures 5 and 6, it appears clearly that the hedge ratios depend on the dependence assumptions. Surprisingly the relative values for $\Delta_{1,0}$ under the 5 scenarios are not always ordered the same. For example, assuming the the financial market behaves as in Scenario 3 (Clayton dependence for each pair), then the deltas obtained with the trivariate Gaussian multivariate distribution can either underestimate or overestimate the hedge ratios $\Delta_{1,0}$.

6 Conclusion

In a dynamic copula setting, it is not clear why the dependence under the objective measure (in the actual world) should be the same as the dependence under the risk-neutral measure. We describe the steps to price a multivariate derivative in this setting and illustrate the study with a dataset of multivariate derivatives prices sold in the US. It is hard to draw firm conclusions from the only data example of this paper. It however provides an illustration of how the pair-copula construction methodology can be applied to model dependence and price multivariate derivatives. We further illustrate that the choice of the dependence has important effects on the pricing of multivariate derivatives and that the Gaussian or the T-Student copula may underprice such derivatives. Finally, note that the empirical analysis also highlights important changes in volatility in the past ten years and therefore the presence of regimes. The price of basket options is very sensitive to the modeling of volatility as well as shifts of regimes. Therefore regime switching models may be more appropriate for pricing long-term derivatives as the ones studied in the paper.

Figure 3: Price of the payoff (22) with maturity 1 year (252 days) when S_1 , S_2 and S_3 are GARCH(1,1) with the parameters as in Table 2, third period, the dependence is as in Table 6 and the annual risk-free rate is 4%. Panel A corresponds to $\tau_{23|1} = 0.1$ and Panel B to $\tau_{23|1} = 0.7$. Prices are calculated in the 6 scenarios presented in Table 6.



Figure 4: Price of the payoff (23) with maturity 1 year (252 days) when S_1 , S_2 and S_3 are GARCH(1,1) with the parameters as in Table 2, third period, the dependence is as in the table 6 and the annual risk-free rate is 4%. Panel A corresponds to $\tau_{23|1} = 0.1$ and Panel B to $\tau_{23|1} = 0.7$. Prices are calculated in the 6 scenarios presented in Table 6.



Figure 5: Delta at time 0 with respect to S_1 of the payoff (22) with maturity 1 year (252 days) when S_1 , S_2 and S_3 are GARCH(1,1) with the parameters as in Table 2, third period, the dependence is as in Table 6 and the annual risk-free rate is 4%. Panel A corresponds to $\tau_{23|1} = 0.1$ and Panel B to $\tau_{23|1} = 0.7$. Prices are calculated in the 6 scenarios presented in Table 6.



Figure 6: Delta at time 0 with respect to S_1 of the payoff (23) with maturity 1 year (252 days) when S_1 , S_2 and S_3 are GARCH(1,1) with the parameters as in Table 2, third period, the dependence is as in the table 6 and the annual risk-free rate is 4%. Panel A corresponds to $\tau_{23|1} = 0.1$ and Panel B to $\tau_{23|1} = 0.7$. Prices are calculated in the 6 scenarios presented in Table 6.



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