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Market value of life insurance contracts under stochastic interest rates and default risk☆

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Abstract

The purpose of this article is to value some life insurance contracts in a stochastic interest rate environment taking into account the default risk of the underlying insurance company. The participating life insurance contracts considered here can be expressed as portfolios of barrier options as shown by Grosen and Jørgensen [J. Risk Insurance 64 (3) (1997) 481–503]. In order to price these options, the Longstaff and Schwartz [J. Finance 50 (3) (1995) 789–820] methodology is used with the Collin-Dufresne and Goldstein [J. Finance 56 (5) (2001) 1929–1957] correction.

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0. Introduction

Life insurance companies offer complex contracts written with the following many covenants: interest rate guarantees, bonus and surrender options, equity-linked policies, choice of a reference portfolio, participating policies. Each particular covenant has a value and is part of the company liabilities. These embedded options should not be ignored and must be priced. Many life-insurance companies, having neglected them for a long time, increased the difficulties they faced in the 1990s.

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Most of the recent studies rely on the Briys and de Varenne (1997a, 1997b) model. These authors aim at providing a fair valuation of liabilities. By this, it is meant that market value is the reference. More precisely, the computed prices must be arbitrage free. The life insurance contracts are thus considered as purely financial assets traded on a liquid market among perfectly informed investors. This fact is taken as a fundamental assumption in these studies, and it is the basic hypothesis we make in this article. Note that this principle is in line with the Financial Accounting Standards Board (FASB) and International Accounting Standard Board (IASB) directives.

Although Briys and de Varenne (1994, 1997a, 1997b) work in continuous time, their model is essentially a single-period one, and furthermore does not take into account the mortality risk. They value the assets and liabilities of an insurance company which sells only one type of contract. The default can occur only at maturity. Their framework is of the Merton type, and they can therefore obtain closed-form formulae which permit to adjust the different parameters involved in a fair contract. Nevertheless, this model can be considered as a prototype in the valuation of life insurance contract.

Miltersen and Persson (2003) propose a multi-period extension and also provide closed form formulae. Bacinello (2001) analyzes the most sold life insurance contract in Italy. She takes into account mortality and suggests a contract which offers the choice among different triplets of technical rate, participation level and volatility. Paying each year a premium, the insured customer gets the guarantee to recover his initial investment accrued at a fixed rate and can possibly benefit from a bonus indexed on a reference portfolio. The pricing is achieved under the standard Black and Scholes model and assuming independence between mortality risk and financial risk.

Tanskanen and Lukkarinen (2003) consider general participating life insurance contracts. Their contract values depend on the evolution of a reference portfolio at different dates. These authors incorporate the following features: minimum interest rate guaranteed each year, right to change each year the reference portfolio, as well as possibility to surrender each year the contract—giving it a Bermudian aspect. They work with constant interest rates and a constant volatility.

Because there are various kinds of contracts and modeling frameworks, the pricing methodologies are diverse. In fact, mortality, a stochastic interest rate environment and stochastic volatilities, for instance, can be taken into account as well as the right to sell back the contract. Participating policies are also multiple. It must be noted that closed form solutions are obtained in the simple Black and Scholes setting. Tanskanen and Lukkarinen (2003) use a numerical procedure to solve their partial differential equation in order to compute the surrender option.

Jørgensen (2001) and Grosen and Jørgensen (2002) show that a life insurance contract with a minimum interest rate guarantee can be expressed in four terms, the final guarantee (equivalent to a zero-coupon bond), the European bonus option associated with a percentage of the positive performance of the company's asset portfolio, if any, a put option linked to the default risk, and finally a fourth term which is a rebate given to the policyholders in case of default prior to the maturity date.

In Grosen and Jørgensen (1997), the possibility of an early payment is envisaged. To treat this American-style contract they use a binomial lattice whereas Jensen et al. (2001) use a finite difference approach. Grosen and Jørgensen (2002) take into account a default barrier of an exponential type. They obtain closed form formulae in the case of constant interest rates. Jørgensen (2001) extends this study to the more difficult case of stochastic interest rates, using a Monte-Carlo approach.

This study is devoted to the valuation of life insurance contracts in the presence of a stochastic term structure of interest rates, it also takes into account the company's default risk. We provide an alternative method to trees, numerical solutions of PDE and Monte-Carlo simulations, schemes usually used to price such contracts. The term structure of interest rates considered here stems from the classical Heath et al. (1992) framework. Amongst the two standard choices of zero-coupon volatilities making the instantaneous risk-free rate Markovian – linear volatility as in the Ho and Lee model or exponential volatility as in the Hull and White model – we take the second one. Our model is therefore a Vasicek one. Note that we could have considered in our paper a full Hull and White or generalized Vasicek framework by relying on a purely exogenously specified (by a set of zero-coupons) initial term structure of interest rates. The extension of our computations to a Ho and Lee choice of zero-coupon volatility is also straightforward. Our valuation method relies on Collin-Dufresne and Goldstein's (2001) article which is an outgrowth of Fortet's (1943) algorithm used by Longstaff and Schwartz (1995) to approximate the first passage time density to a given level by a log-normal process.

Firstly, we give the general setting of our model. Then we detail the adopted methodology, and finally we present some numerical applications giving the market price of our life insurance contract and we explain how to choose the parameters leading to a fair value contract.

1. Framework

We want to show how to price a participating life insurance contract with a minimum guaranteed rate in presence of default risk of the issuing company. We begin with the definition of the contract and the default process, before concluding with the modeling of the interest rate process.

1.1. Contract and default model

We consider an insurance company with two types of agents: policyholders and shareholders. The policyholders possess the same unique contract which will be defined precisely in the following. The considered life-insurance company has no debt and its planning horizon is finite with T as maturity, being also the expiry date of the contract. Let A_0 be the assets initial value, $L_0 = \alpha A_0$ the initial investment by policyholders, and $E_0 = (1-\alpha)A_0$ is the initial equity.

The policyholder is guaranteed a fixed interest rate r_g. So, the guaranteed amount at T is a priori $L_T^g = L_0\,\mathrm{e}^{r_g T}$. However, when the firm defaults, this amount will be lowered, on the contrary it will be raised if exceptional results of the company occur. The next step is to express these payments according to the firm's assets dynamics. We refer to a continuous time economy with a perfect financial market into which our life insurance company is included.

1.2. Payment at maturity

Let us look at what happens at T: if $A_T \geq L_T^g$, the company is able to fulfill its commitments, otherwise $A_T < L_T^g$ and it is insolvent. In this case, policyholders receive A_T and equityholders nothing. Because we assume a participating policy, when the assets generate value such that $A_T > L_T^g/\alpha$ with $\alpha < 1$, the policyholder is given a bonus, say δ, a contractual part of the surplus, known as the participation coefficient. To sum up, policyholders receive at T, assuming no prior bankruptcy:

$$\Theta_L(T) = \begin{cases} A_T & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \leq A_T \leq \dfrac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \dfrac{L_T^g}{\alpha} \end{cases}$$

In this paragraph we have mimicked the Merton (1974) default approach. We can rewrite the payoff in a more concise form:

$$\Theta_L(T) = L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+ \tag{1}$$

The first term is the promised amount, the second term – called "bonus option" – is linked to the participating clause, the third one is a put option associated with the default risk.

These last payoffs share the same features as usual European options. According to our fundamental hypothesis and assuming that the assets dynamics follows a geometric Brownian motion it is easy to price them. For more details and closed form solutions, we refer to Briys and de Varenne (1994).

1.3. Company early default

Now we assume that default can occur prior to the maturity T. The default mechanism we choose is of a structural type, so we introduce an activating barrier on the firm's assets. From now on, bankruptcy can occur at any time t before T. The contract value depends on the assets price before the expiry of the contract and not only on their price at T. The barrier is chosen exponential and is denoted by B_t.

The firm pursues its activities until T if:

$$\forall t \in [0, T[, \quad A_t > \lambda L_0 \, \mathrm{e}^{r_g t} \triangleq B_t \tag{2}$$

If it is not so, it is declared bankrupt. Let τ be the default time; it is the first time when A_t hits the barrier B_t, otherwise stated:

$$\tau = \inf\{t \in [0, T]/A_t < B_t\} \tag{3}$$

With λ greater than 1, the firm is able, even when going bankrupt, to pay back policyholders their investments accrued at the guaranteed rate r_g. The residual capital (equal to $(\lambda - 1)L_0 \, \mathrm{e}^{r_g \tau}$) can be used to pay bankruptcy costs or can be distributed to shareholders. The situation $\lambda \geq 1$ is therefore very favorable to policyholders and regulators. Theoretically it is a risk free position. On the contrary, in the case when $\lambda < 1$, the firm is totally insolvent in the case of bankruptcy and unable to meet its commitments.

So, policyholders will receive in case of early default:

$$\Theta_L(\tau) = \begin{cases} L_0 \, \mathrm{e}^{r_g \tau} & \text{if } \lambda \geq 1 \\ \lambda L_0 \, \mathrm{e}^{r_g \tau} & \text{if } \lambda < 1 \end{cases} = \min(\lambda, 1) L_0 \, \mathrm{e}^{r_g \tau} = \min(\lambda, 1) L_\tau^g \tag{4}$$

1.4. Contract value

Using the standard machinery of arbitrage theory in continuous time and denoting by Q the risk-neutral probability measure, the arbitrage free price of our life insurance contract (hereafter LIC) at time t can be written as:

$$V_L(t) = \mathbb{E}_Q^t[\mathrm{e}^{-\int_t^T r_s \, \mathrm{d}s}[L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+]\mathbb{1}_{\tau \geq T} + \mathrm{e}^{-\int_t^\tau r_s \, \mathrm{d}s} \min(\lambda, 1) L_\tau^g \mathbb{1}_{\tau < T}] \tag{5}$$

This contract can be split up into four simpler subcontracts:

$$V_L(t) = \widehat{\mathrm{GF}}_t + \widehat{\mathrm{BO}}_t - \widehat{\mathrm{PO}}_t + \widehat{\mathrm{LR}}_t \tag{6}$$

where $\widehat{\mathrm{GF}}$ corresponds to the final guarantee, $\widehat{\mathrm{BO}}$ stands for the "bonus option", $\widehat{\mathrm{PO}}$ for the default put on which policyholders are short, and, at last, $\widehat{\mathrm{LR}}$ is the rebate paid to policyholders in case of early default. Individually these four subcontracts can be written as:

$$\begin{aligned} &\widehat{\mathrm{GF}}_t = \mathbb{E}_Q^t[\mathrm{e}^{-\int_t^T r_s \, \mathrm{d}s} \mathbb{1}_{\tau \geq T} L_T^g], \qquad \widehat{\mathrm{BO}}_t = \mathbb{E}_Q^t[\mathrm{e}^{-\int_t^T r_s \, \mathrm{d}s} \mathbb{1}_{\tau \geq T} \delta(\alpha A_T - L_T^g)^+], \\ &\widehat{\mathrm{PO}}_t = \mathbb{E}_Q^t[\mathrm{e}^{-\int_t^T r_s \, \mathrm{d}s} \mathbb{1}_{\tau \geq T}(L_T^g - A_T)^+], \qquad \widehat{\mathrm{LR}}_t = \mathbb{E}_Q^t[\mathrm{e}^{-\int_t^\tau r_s \, \mathrm{d}s} \mathbb{1}_{\tau < T} \min(\lambda, 1) L_\tau^g] \end{aligned} \tag{7}$$

Note that closed form formulae are available with constant interest rates (see Grosen and Jørgensen, 2002). Our aim in this article is to value our LIC in a reasonably sound stochastic interest rate environment. Of course this problem is rather complex and will lead us to semi-closed formulae. Let us now turn back to the term structure of interest rate.

1.5. Assets dynamics and interest rate modeling

The most efficient way to price options in a stochastic interest rates environment is to use the change of numéraire technique and to choose an ad hoc zero-coupon bond as new numéraire. So, forward-neutral probability measures technically play a key role. We need to know the it T-forward-neutral assets dynamics as well as the dynamics of a default free zero-coupon bond with expiry date T. We denote by $P(t, T)$ its price at current time t. We assume that the assets price follows a geometric Brownian motion in the risk-neutral world and we use a one factor Heath et al. (1992) interest rate model with a deterministic volatility for the T-zero-coupon bond of an exponential type (this is the Hull and White choice). With $\nu > 0$ and $a > 0$, the volatility structure can be written as follows:

$$\sigma_P(t,T) = \frac{\nu}{a}(1 - \mathrm{e}^{-a(T-t)}) \tag{8}$$

In this case, the dynamics of the instantaneous interest rate r under the forward-neutral probability Q_T can be written like:

$$\mathrm{d}r_t = a(\theta_t - r_t)\,\mathrm{d}t + \nu\,\mathrm{d}Z_1^{\mathrm{Q}_T}(t) \tag{9}$$

where $\theta_t = \theta - (\nu^2/a^2)(1 - \mathrm{e}^{-a(T-t)})$.

Under the risk-neutral probability measure Q, the assets value, A_t, and the zero-coupon bond price with expiry date T, $P(t, T)$, follow the stochastic diffusions

$$\frac{\mathrm{d}A_t}{A_t} = r_t\,\mathrm{d}t + \sigma\,\mathrm{d}Z^{\mathrm{Q}}(t) \tag{10}$$

and

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r_t\,\mathrm{d}t - \sigma_P(t,T)\,\mathrm{d}Z_1^{\mathrm{Q}}(t)$$

where $Z^{\mathrm{Q}}(t)$ and $Z_1^{\mathrm{Q}}(t)$ are Q-standard Brownian motions. Let ρ be the correlation coefficient between these two Brownian movements ($\mathrm{d}Z^{\mathrm{Q}}\,\mathrm{d}Z_1^{\mathrm{Q}} = \rho\,\mathrm{d}t$).

Let us now consider a Brownian motion Z_2^{Q} independent from Z_1^{Q} (such that $\mathrm{d}Z_1^{\mathrm{Q}}\,\mathrm{d}Z_2^{\mathrm{Q}} = 0$); the Brownian motion Z^{Q} can be expressed as

$$\mathrm{d}Z^{\mathrm{Q}}(t) = \rho\,\mathrm{d}Z_1^{\mathrm{Q}}(t) + \sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{Q}}(t)$$

In this way we decorrelate the interest rate risk from the firm assets risk. The assets dynamics (10) then writes:

$$\frac{\mathrm{d}A_t}{A_t} = r_t\,\mathrm{d}t + \sigma\rho\,\mathrm{d}Z_1^{\mathrm{Q}}(t) + \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{Q}}(t) \tag{11}$$

Let us now denote by Q_T the T-forward-neutral measure. It is defined through its Radon-Nikodym derivative

$$\frac{\mathrm{dQ}_T}{d\mathrm{Q}} = \mathrm{e}^{-\int_0^T \sigma_P(s,T)\,\mathrm{d}Z_1^{\mathrm{Q}}(s) - (1/2)\int_0^T \sigma_P^2(s,T)\,\mathrm{d}s}$$

From Girsanov theorem the process $Z_1^{\mathrm{Q}_T}$ defined by $\mathrm{d}Z_1^{\mathrm{Q}_T} = \mathrm{d}Z_1^{\mathrm{Q}} + \sigma_P(t,T)\,\mathrm{d}t$ is a Q_T-Brownian motion. The process $Z_2^{\mathrm{Q}_T}$ is then built such that $Z_1^{\mathrm{Q}_T}$ and $Z_2^{\mathrm{Q}_T}$ are Q_T-non correlated standard Brownian motions. Under Q_T the prices $P(t, T)$ and A_t follow the stochastic differential equations

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))\,\mathrm{d}t - \sigma_P(t,T)\,\mathrm{d}Z_1^{Q_T}$$

and

$$\frac{\mathrm{d}A_t}{A_t} = (r_t - \sigma\rho\sigma_P(t,T))\,\mathrm{d}t + \sigma(\rho\,\mathrm{d}Z_1^{Q_T} + \sqrt{1-\rho^2}\,\mathrm{d}Z_2^{Q_T}) \tag{12}$$

After integration, one obtains

$$A_t = \frac{A_0}{P(0,t)}\exp\left(\int_0^t (\sigma_P(u,t)+\rho\sigma)\,\mathrm{d}Z_1^{Q_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{Q_T}(u) + \int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t)+\rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2}\right)\mathrm{d}u\right) \tag{13}$$

This formula will be useful to simulate the process A_t as well as to study the moments of $\ln(A_T)$; we shall see next that it is a prerequisite to solve our problem.

1.6. The valuation

We now present the valuation of our LIC under the setting defined above. For the sake of simplicity, we set the current time to zero ($t = 0$). Using the fact that the relative prices are martingale under the T-forward-neutral equivalent martingale measure, we can rewrite formula (5) according to:

$$V_L(0) = P(0,T)\mathbb{E}_{Q_T}[(L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+)\mathbb{1}_{\tau\geq T} + \mathrm{e}^{\int_\tau^T r_s\,\mathrm{d}s}\min(\lambda,1)L_\tau^g\,\mathbb{1}_{\tau<T}]$$

Using the relation $\mathbb{1}_{\tau\geq T} = 1 - \mathbb{1}_{\tau<T}$, the expression of the subcontracts in (7) lead in a very simple way in the T-forward-neutral-universe to:

$$V_L(0) = P(0,T)(\mathrm{GF} + \mathrm{BO} - \mathrm{PO} + \mathrm{LR}) \tag{14}$$

where

$$\mathrm{GF} = L_T^g(1-E_1), \qquad \mathrm{BO} = \alpha\delta(E_7 - E_2) - \delta L_T^g(E_8 - E_3),$$
$$\mathrm{PO} = L_T^g(E_9 - E_4) - E_{10} + E_5, \qquad \mathrm{LR} = \min(\lambda,1)L_0\,E_6 \tag{15}$$

and where we introduce the following quantities

$$
\begin{array}{ll|ll}
E_1 = \mathrm{Q}_T\,[\tau < T] & & E_6 = \mathbb{E}_{\mathrm{Q}_T}\left[e^{\int_\tau^T r_s ds} e^{r_g \tau} \mathbb{1}_{\tau<T}\right] & \\
E_2 = \mathbb{E}_{\mathrm{Q}_T}\left[A_T \mathbb{1}_{\left\{A_T > \frac{L_T^g}{\alpha}\ ,\ \tau<T\right\}}\right] & & E_7 = \mathbb{E}_{\mathrm{Q}_T}\left[A_T \mathbb{1}_{A_T > \frac{L_T^g}{\alpha}}\right] & \\
E_3 = \mathrm{Q}_T\left[A_T > \frac{L_T^g}{\alpha}\ ,\ \tau < T\right] & & E_8 = \mathrm{Q}_T\left[A_T > \frac{L_T^g}{\alpha}\right] & \\
E_4 = \mathrm{Q}_T[A_T < L_T^g\ ,\ \tau < T] & & E_9 = \mathrm{Q}_T[A_T < L_T^g] & \\
E_5 = \mathbb{E}_{\mathrm{Q}_T}\left[A_T \mathbb{1}_{A_T < L_T^g} \mathbb{1}_{\tau<T}\right] & & E_{10} = \mathbb{E}_{\mathrm{Q}_T}\left[A_T \mathbb{1}_{A_T < L_T^g}\right] & (16)
\end{array}
$$

(16)

In the next section, we show how to compute these expressions. The ones in which the default time τ does not intervene lead to closed form formulae. For the others, as far as we know, closed form formulae are not available, hence, in order to compute them, we use an approximation of the distribution of τ. This is the object of the following paragraph and constitutes the core of our pricing methodology.

2. Valuation methodology

To price our LIC we need to compute each expectation E_i in (16). We have to know the law of the default time τ—first passage time of the lognormal process of the assets A_t at the exponential barrier, given in (2).

Longstaff and Schwartz (1995) use Fortet's result (1943) to approximate the law of τ in a similar problem to ours: the pricing of defaultable bonds and defaultable floating rate notes. However, the Longstaff and Schwartz (1995) approximation is not satisfactory and mathematically not valid. Collin-Dufresne and Goldstein (2001) brought a correction to the previous approximation which validates the method for problems of the kind we encounter. We call this corrected method the extended Fortet's method. It is the key solution to the pricing of our LIC in this article; let us now explain this method.

Firstly, we adopt the following convention: $l_t = \ln(\chi_t) = \ln(A_t) - r_g t$. For this process, the default barrier becomes $h = \ln(\lambda L_0)$; we assume it is below l_0, the initial value of the process under study. Besides, it can be shown that the process l_t obeys under Q_T the following stochastic differential equation (applying Itō's lemma to Eq. (12)):

$$
\mathrm{d}l_t = \left(r_t - r_g - \frac{\sigma^2}{2} - \sigma\rho\sigma_P(t,T)\right)\mathrm{d}t + \sigma\rho\,\mathrm{d}Z_1^{\mathrm{Q}_T} + \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{Q}_T}
$$

So, we have to study the first passage time of l_t to the constant level h, put more explicitly

$$
\tau = \inf\{t \in [0, T] / l_t \leq h\}
$$

In order to compute the expectations in formula (14), we choose to approximate the law of τ.

Let p_τ be the density of the random variable τ at time t under the T-forward-neutral measure Q_T, when the interest rate has value r_t and $l_t = h$. We will calculate it as a piecewise constant function. The approximation consists of a time and interest rate discretization. The interval $[0, T]$ is subdivided into n_T subperiods of length $\delta_t = T/n_T$. The interest rate is subdivided between $r_{\min}$ and $r_{\max}$ into n_r intervals with the same size $\delta_r = (r_{\max} - r_{\min})/n_r$. At last, let us define by $t_j = j\delta_t$ and $r_i = r_{\min} + i\delta_r$ the discretized values of time and interest rate.

We give a recursive approximation of the density of τ as a piecewise constant function on $[t_j, t_{j+1}]$ when the interest rate is between r_i and r_{i+1}. We denote this density by

$$p(r_i, t_j), \quad j = 0, \dots, n_T - 1, \; i = 0, \dots, n_r.$$

Next, we need to compute the probability of the event $\tau \in [t_j, t_{j+1}]$ with $r \in [r_i, r_{i+1}]$; it expresses as

$$q(i, j) = \delta_t \delta_r p(r_i, t_j)$$

Let $f(l_t, r_t, t | l_s = a, r_s = r, s)$ be the conditional law of (l_t, r_t) given $\{l_s = a, r_s = r\}$.

Define respectively Φ, Ψ and g by:

$$\Phi(r_t, t) = \int_{-\infty}^{h} f(l_t, r_t, t | l_0, r_0, 0)\, \mathrm{d}l_t,$$

$$\Psi(r_t, t | r_s, s) = \int_{-\infty}^{h} f(l_t, r_t, t | l_s = h, r_s, s)\, \mathrm{d}l_t, \qquad g(r_s, s) = p_\tau(l_s = h, r_s, s | l_0, r_0, 0)$$

It can be shown (for further details, see Collin-Dufresne and Goldstein) that the quantities $q(i, j)$ may be computed by a recursive algorithm. First, the quantities $q(i, 0)$ are computed for every i:

$$q(i, 0) = \Phi(r_i, t_0)$$

from them the quantities $q(i, j)$ for $j \geq 2$ are recursively obtained:

$$q(i, j) = \Phi(r_i, t_j) - \sum_{v=0}^{j-1} \sum_{u=0}^{n_r} q(u, v) \Psi(r_i, t_j | r_u, t_v) \tag{17}$$

To calculate $q(i, j)$, the expressions $\Phi(r_t, t)$ and $\Psi(r_t, t | r_s, s)$ are needed. Since the processes l_t and r_t are Gaussian, the conditional law of l_t given the σ-tribe generated by the information available at time s and given r_t, is Gaussian, with mean $\mu(r_t, l_s, r_s)$ and variance $\Sigma^2(r_t, l_s, r_s)$. The computations and results are given in Appendix A.1 as well as the centered moments of order 1 and 2 of the processes l_t and r_t.

Let us denote, as usual, by $\mathcal{N}$ the cumulative distribution function of the standard normal law. Using the previous Gaussian conditional law and the Bayes' rule we obtain

$$\Phi(r_t, t) = f_r(r_t, t | l_0, r_0, 0) \mathcal{N}\left(\frac{h - \mu(r_t, l_0, r_0)}{\sqrt{\Sigma^2(r_t, l_0, r_0)}} \right),$$

$$\Psi(r_t, t | r_s, s) = f_r(r_t, t | l_s = h, r_s, s) \mathcal{N}\left(\frac{h - \mu(r_t, l_s = h, r_s)}{\sqrt{\Sigma^2(r_t, l_s = h, r_s)}} \right)$$

where we have an explicit formula for the transition density f_r of r (which is a Gaussian process):

$$f_r(r_t, t | l_s = h, r_s, s) = \frac{1}{\sqrt{2\pi v}} \mathrm{e}^{-((r_t - m)^2 / 2v)}$$

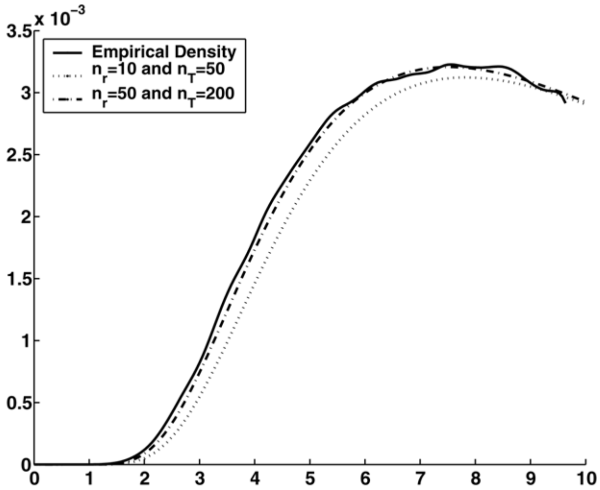

Fig. 1. Empirical and extended Fortet's approximate density.

where $m = \mathbb{E}[r_t|r_s]$ and $v = \mathrm{Var}[r_t|r_s]$, respectively, stand for the conditional moments of r_t given r_s. They are also provided in Appendix A.1.

To sum up, we have now, with formula (17) the possibility to compute the $q(i, j)$ terms, which give us the density of τ we were looking for. Now we are equipped to obtain our expectations.

Fig. 1 illustrates the fact that this corrected method gives an approximated density $p_j = \sum_i q(i, j)$ which adjusts satisfactorily the empirical density of τ (obtained here by Monte-Carlo simulation). The extended Fortet's method is, of course, more time-consuming than the ordinary Fortet's method, because of the double discretization; however it is far less time consuming than Monte-Carlo simulations.

2.1. The quasi-closed form formula for the LIC

At present, we have to apply our method to compute the expectations in (16) in order to get $V_L(0)$. Each term involving τ is computed using the extended Fortet's method.

For this goal, we need to know precisely the moments of l_t (the formulae in (16), which are expressed as functions of A, can indeed be rewritten as functions of l) and the moments of r_t and also the conditional moments of l_t given r_t. These calculations are provided in Appendix A.1.

Let us begin with the computation of E_1. From its definition, it can be written in the following integral form:

$$\mathrm{Q}_T[\tau < T] = \int_0^T \int_{-\infty}^{+\infty} p_\tau(l_s = h, r_s, s|l_0, r_0, 0)\,\mathrm{d}r_s\,\mathrm{d}s$$

We then discretize according to time and rate and replace the exact density p_τ of τ by its approximation $q(i, j)$:

$$E_1 = \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} q(i, j)$$

We also detail the computation of E_2, the other approximated E_i will be obtained in a similar manner.

$$E_2 = \mathbb{E}_{Q_T}\left[A_T\, e^{-r_g T}\, e^{r_g T}\mathbb{1}_{\{A_T\, e^{-r_g T} > L_0/\alpha,\tau<T\}}\right] = \mathbb{E}_{Q_T}\left[\chi_T\, e^{r_g T}\mathbb{1}_{\chi_T > L_0/\alpha}\mathbb{1}_{\tau<T}\right]$$
$$= e^{r_g T}\,\mathbb{E}_{Q_T}[e^{l_T}\mathbb{1}_{l_T>\ln(L_0/\alpha)}\mathbb{1}_{\tau<T}]$$

By conditioning (being here the key tool) we obtain:

$$E_2 = e^{r_g T}\int_0^T \mathrm{d}s\int_{-\infty}^{+\infty}\mathrm{d}r_s\, g(r_s, s)\mathbb{E}_{Q_T}\left[e^{l_T}\mathbb{1}_{\{l_T>\ln(L_0/\alpha)\}}|l_s = h, r_s, s, \tau = s\right]$$

In this last formula, the expectation only concerns l_T. But we do not know the density of l_T, we only know the conditional law of l_T given r_T, and the transition density of an Ornstein–Uhlenbeck process, denoted by f_r. Therefore:

$$E_2 = e^{r_g T}\int_0^T \mathrm{d}s\int_{-\infty}^{+\infty}\mathrm{d}r_s\, g(r_s, s)\int_{-\infty}^{+\infty}\mathrm{d}r_T\, f_{r|\mathcal{F}_s}(r_T)\,\mathbb{E}_{Q_T}\left[e^{l_T}\mathbb{1}_{\{l_T>\ln(L_0/\alpha)\}}|r_T, \mathcal{F}_s\right]$$

The law of l_T conditional on $\mathcal{F}_s$ and given r_T is Gaussian ; its first two centered moments are $\hat{\mu}_{s,T} = \mu(r_T, l_s, r_s)$ and $\hat{\Sigma}^2_{s,T} = \Sigma^2(r_T, l_s, r_s)$.

Let X be the Gaussian random variable $\mathcal{N}(m, \sigma^2)$, we define:

$$\Phi_1(m; \sigma; a) = \mathbb{E}[e^X\mathbb{1}_{e^X>a}] = \exp\left(m + \frac{\sigma^2}{2}\right)\mathcal{N}\left(\frac{m+\sigma^2-\ln(a)}{\sigma}\right)$$

The expectation E_2 can be rewritten as:

$$E_2 = e^{r_g T}\int_0^T \mathrm{d}s\int_{-\infty}^{+\infty}\mathrm{d}r_s\, g(r_s, s)\int_{-\infty}^{+\infty}\mathrm{d}r_T\, f_r(r_T|r_s, s, l_s)\Phi_1\left(\hat{\mu}_{s,T}; \hat{\Sigma}_{s,T}; \frac{L_0}{\alpha}\right)$$

Then, the extended Fortet's approximation for E_2 is:

$$E_2 = e^{r_g T}\sum_{j=0}^{n_T}\sum_{i=0}^{n_r}\sum_{k=0}^{n_r}\delta_r f_r(r_k|r_i, t_j, l_{t_j})\Phi_1\left(\hat{\mu}_{t_j,T}; \hat{\Sigma}_{t_j,T}; \frac{L_0}{\alpha}\right)q(i, j)$$

With the same scheme, we give the formulae for the others E_i given in (16). It can be shown that

$$E_3 = e^{r_g T}\sum_{j=0}^{n_T}\sum_{i=0}^{n_r}\sum_{k=0}^{n_r}\delta_r f_r(r_k|r_i, t_j, l_{t_j})\mathcal{N}\left(\frac{\hat{\mu}_{t_j,T} - \ln\left(\frac{L_0}{\alpha}\right)}{\sqrt{\hat{\Sigma}^2_{t_j,T}}}\right)q(i, j)$$

and

$$E_4 = e^{r_g T}\sum_{j=0}^{n_T}\sum_{i=0}^{n_r}\sum_{k=0}^{n_r}\delta_r f_r(r_k|r_i, t_j, l_{t_j})\mathcal{N}\left(\frac{\ln(L_0) - \hat{\mu}_{t_j,T}}{\sqrt{\hat{\Sigma}^2_{t_j,T}}}\right)q(i, j)$$

For the computation of E_5, we define

$$\Phi_2(m;\sigma;a) = \mathbb{E}[e^X \mathbb{1}_{e^X<a}] = \exp\left(m+\frac{\sigma^2}{2}\right)\mathcal{N}\left(\frac{\ln(a)-m-\sigma^2}{\sigma}\right)$$

where X is a Gaussian r.v. $\mathcal{N}(m,\sigma^2)$.

We then obtain

$$E_5 = e^{r_g T}\sum_{j,i,k}\delta_r f_r(r_k|r_i,t_j,l_{t_j})\Phi_2(\hat{\mu}_{t_j,T};\hat{\Sigma}_{t_j,T};L_0)q(i,j)$$

and

$$E_6 = \sum_{j=0}^{n_T}\sum_{i=0}^{n_r} e^{r_g t_j}\mathbb{E}_{Q_T}\left[e^{\int_{t_j}^T r_u\,du}|r_{t_j}=r_i,t_j,l_{t_j}=h\right]q(i,j)$$

To compute E_6, we use the fact that under Q_T, $\int_{t_j}^T r_u\,du$ follows a Gaussian law whose parameters are given at the end of Appendix A.2. Finally E_6 is obtained thanks to a classical property of Gaussian random variables: if Y follows $\mathcal{N}(m,\sigma^2)$ then $\mathbb{E}[e^Y] = e^{m+\sigma^2/2}$.

In expectations E_7, E_8, E_9, and E_{10}, the random time τ does not intervene. Furthermore the random variable χ_T is lognormal with moments M_T and V_T (computed in Appendix A.1). Hence, explicit formulae for the last four expectations can be obtained.

Indeed, applying the properties associated with the functions Φ_1 and Φ_2, we obtain

$$\begin{array}{l|l} E_7 = e^{r_g T}\,\Phi_1\left(M_T;\sqrt{V_T};\frac{L_0}{\alpha}\right) & E_8 = \mathcal{N}\left(\frac{M_T-\ln\left(\frac{L_0}{\alpha}\right)}{\sqrt{V_T}}\right) \\ E_9 = \mathcal{N}\left(\frac{\ln(L_0)-M_T}{\sqrt{V_T}}\right) & E_{10} = e^{r_g T}\,\Phi_2\left(M_T;\sqrt{V_T};L_0\right) \end{array} \tag{18}$$

To sum up, in order to compute the different E_i, we need to know Φ_1, Φ_2, f_r, and the different moments given above which are made explicit in Appendix A.1, as well as the probabilities $q(i,j)$. To obtain accurate results, it is sufficient to use a grid with a thin mesh, which can be done with n_T and n_r large enough.

3. Numerical analysis

In this section we make a numerical analysis on our LIC. Two parameters will happen to play a key role: the guaranteed rate r_g and the participating coefficient δ.

The parameters r_g and δ cannot be fixed arbitrarily. The guaranteed rate must be neither too high (bankruptcy risk would be too high in the case of falling interest rates), nor too low (unfavorable contract to policyholders). Besides, it cannot go beyond a legal threshold limit. In France, this threshold is typically around 2.75%. As far as the participating level is concerned, it is calculated such that LICs be fair to both sides. The participating level necessarily varies contrary to the guaranteed rate: the higher the former the lower the latter and vice versa.

There are infinitely many couples (δ, r_g) leading to a fair contract. These parameters depend, of course, on the company's investment policy. That is to say, in our model, they depend on the assets volatility σ and on the default barrier level λ. However, all these contracts are not acceptable, δ must be between 0 and 1. Besides, the participating

Table 1
Data

A_0	a	ν	θ	r_0	ρ	σ	T	λ	α
100	0.4	0.008	0.06	0.03	-0.02	0.1	10	0.8	0.85

Table 2
Contract and subcontracts values

Extended Fortet	BO	GF	PO	LR	Contract	Time (min)
$n_T = 200, n_r = 50$	34.428	99.226	0.119	10.160	84.990	2
$n_T = 500, n_r = 50$	34.428	99.197	0.115	10.193	84.9995	10

coefficient must obey legal constraints ; for example, δ must be greater than 85% in France (cf. Briys and de Varenne, 1997b).

As a first step, we recapitulate the values we choose for the parameters involved in our study. We then turn to the numerical valuation of our LICs and make a comparison of the extended Fortet's method and Monte-Carlo simulations. We also show how to calculate the participating level. Finally, we conclude this section by a sensitivity analysis of the contact price to the assets volatility.

3.1. Data

We give in Table 1 the chosen parameters values. Some will be changed after, in particular volatility σ and barrier level λ.

Recall that A_0 stands for the initial assets value of our company, a, ν, θ and r_0 determine the instantaneous interest rate process, and ρ is the correlation coefficient between the assets process and the instantaneous interest rate process. The small value for σ, set to 10%, corresponds to a standard investment (approximately half in stocks and half in bonds) by the considered life insurance company. Finally, the contract maturity T is set equal to 10 years, and α is the initial proportion of investment by the insured on the total liabilities of the firm.

3.2. Numerical results

We now examine in the following the numerical results we could obtain for the contract value and the fair participating level.

3.2.1. Contract valuation

Tables 2 and 3 display the LIC contract and subcontracts numerical estimations, done with the extended Fortet and Monte-Carlo methods, respectively, using the parameters defined in the previous subsection and taking $r_g = 2.6\%$ and $\delta = 90.23\%$. Five million sample paths have been used in Monte-Carlo simulations for each valuation.

The first remark we must emphasize on is that the extended Fortet method is by far faster than the Monte-Carlo method. Ten minutes of computation time is not instantaneous (as is the case with a closed form formula) but is extremely efficient in the numerical valuation of a complex contract submitted to both interest rate risk and default risk.

Table 3
Contract and subcontracts values

Monte-Carlo	BO	GF	PO	LR	Contract	Time
Step = 1/12	34.10	100.6	0.41	8.87	84.6	20 min
Step = 1/52	34.14	99.87	0.38	9.57	84.7	1 h 30 min
Step = 1/365	34.20	99.22	0.29	10.16	84.8	1 day

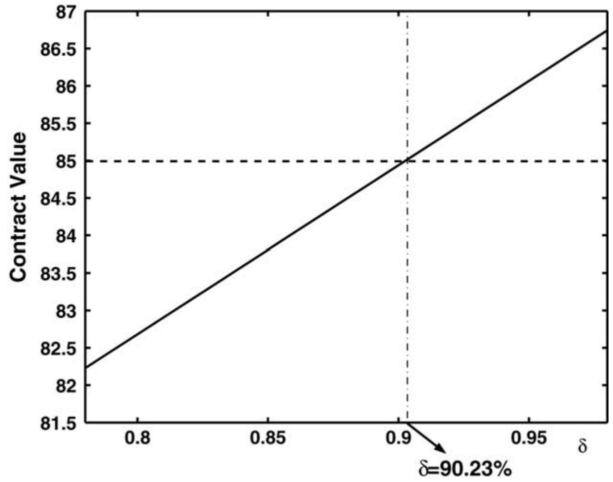

Fig. 2. Contract value (w.r.t. δ).

Furthermore, we observe rather rapidly a convergence for the contract and subcontracts prices when using the extended Fortet's method, while Monte-Carlo converges poorly for some subcontracts such that the default put PO. Hence to obtain a sufficient precision with Monte-Carlo, it would be necessary to launch simulations lasting many days, which is unacceptable for practical use.

Our numerical experiments show, as confirmed by Monte-Carlo simulations, that the prices obtained with the extended Fortet's method are reliable, and in a quite short computation time. The contract fair value is 85 and the extended Fortet's method provides an accuracy of three digits in 10 min. On the contrary, the Monte-Carlo method is very slow in converging: indeed our path-dependent problem requires a very fine discretization (many time steps) for each – amongst many – sample path. The implementation of both methods has been done with an extensive use of Matlab vectorization tools, on a 3 GHz computer.

3.2.2. Computation of the participating level

We are looking for participating levels fair to both policyholders and the company. This will be done under the following equilibrium condition: a contract is said to be fair if the policyholders' initial investment $L_0 = \alpha A_0$ is equal to the total value of subscribed contracts.

We present in Figs. 2 and 3 the contract value as a function of δ and the guaranteed rate r_g. Note that the level 85 corresponds to the L_0 value. Fig. 2 is obtained with a guaranteed rate set at 2.6%. The higher the participating level, the higher the contract value. Let us note that only one value of δ corresponds to a fair value contract, which is the initial investment L_0. Fig. 3 is obtained with a participating level set to $\delta = 0.9023$, and represents the contract value as a function of the guaranteed rate. Here again, only one value of r_g leads to a fair contract.

Now, let us explain the employed procedure. If one wants to determine the participating coefficient with a given guaranteed rate, one has to compute:

$$\delta = \frac{\frac{L_0}{P(0,T)} - \mathrm{GF} + \mathrm{PO} - \mathrm{LR}}{\alpha(E_7 - E_2) - L_T^g(E_8 - E_3)}$$

The calculation of the guaranteed rate given the participating coefficient is more difficult. One has to use a root searching algorithm with the constraint that the contract initial value is equal to L_0.

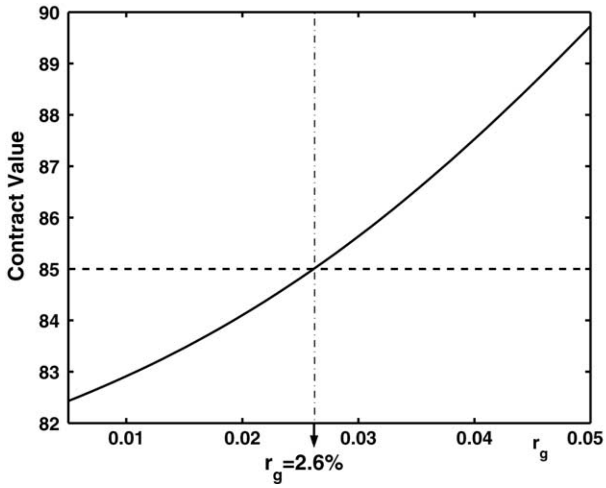

Fig. 3. Contract value (w.r.t. r_g).

3.3. Sensitivity to volatility

We examine now the sensitivity of the participating level δ – and guaranteed rate r_g – to the assets volatility. These sensitivities are displayed in Figs. 4 and 5.

Fig. 4 shows that the weaker the participating level is, the more it is necessary to compensate with a big guaranteed interest rate. On the graph the curves are presented in descending order with respect to δ.

On the opposite, we remark in Fig. 5 that (fair participating curves are presented in descending order with respect to r_g) a low guaranteed rate must be compensated by a high level of the participating coefficient.

Let us examine now the impact of volatility. It is clear from Fig. 5 that the guaranteed rate begins to fall before moving up as volatility increases. When the volatility σ is low, the default risk is negligible, a rising volatility corresponds to a rising return. Given a fixed participating coefficient δ, the guaranteed rate must necessarily decrease to preserve a fair contract. Should r_g remain constant, the contract would be more and more advantageous when

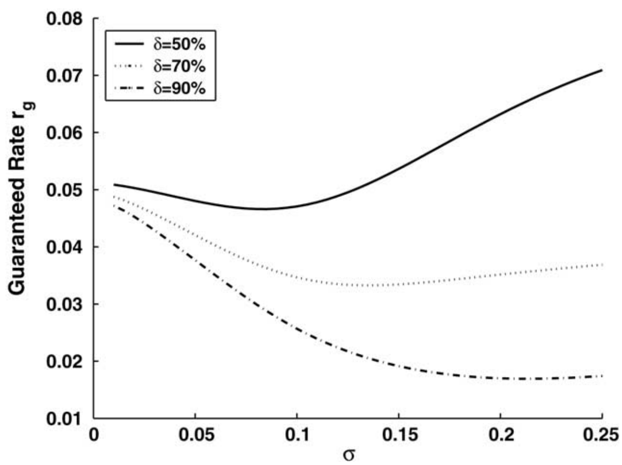

Fig. 4. r_g as a function of σ.

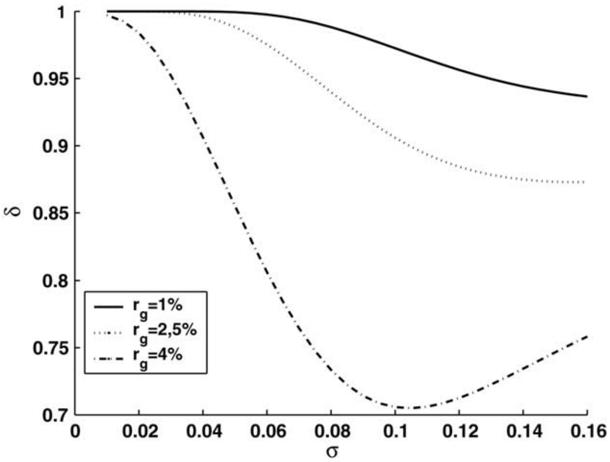

Fig. 5. δ as a function of σ.

σ increases. On the contrary, when σ is above 10%, the default risk becomes important and the probability for policyholders to get back their guaranteed investment diminishes; it is then necessary to compensate with a higher guaranteed rate.

At last, let us analyze Fig. 5. Here again we have a similar behavior. With a fixed guaranteed rate, the participating level begins to decrease before rising, as long as volatility increases. When the volatility is low, in other words, when we can consider that no default risk exists, a volatility rise implies a better return; in order to limit the policyholders advantage, the participating level must decrease. On the contrary, when volatility is high, default risk is important, and necessarily the participating level has to be raised up, given the guaranteed rate, to preserve fairness (policyholders bearing the risk not to recover their initial investment).

4. Conclusion

In this article we have proposed a new method to value typical participating life insurance contracts, with minimum guaranteed rate, in the presence of default risk, and in a stochastic interest rate environment. We have determined the fair participating level, which is a delicate and important point for a life insurance company. We have also analyzed the sensitivity of the main parameters to volatility.

The suggested method relies on Fortet's equation (1943) giving the first passage time of the assets process to the default barrier, and consequently paving the way for computing diverse exotic options embedded in the contract involving this random time. This method has been used in Finance for the first time by Longstaff and Schwartz (1995) then by Collin-Dufresne and Goldstein (2001). These last authors have amended the Longstaff and Schwartz approach extending it in a rigorous way to two dimensional continuous Markov processes. It is this method we used under the name of extended Fortet's method.

Confronting with Monte-Carlo method, we have proved that the extended Fortet's method performs very well to value typical life insurance contracts in a rather general context. More than that, the extended Fortet's method permits to value these contracts in a very fast computing-time, which constitutes certainly a convincing argument for practioners.

Because the fair participating coefficient asks for a root searching algorithm, it is important to have a rapid and efficient method to value LICs. Once again one can perceive the advantage of the proposed method with respect to Monte-Carlo simulations routinely used.

Appendix A

A.1. Moments and conditional moments of χ_T

Recall that the process χ is defined by $\ln(\chi_t) = \ln(A_t) - r_g t$. For a fixed t, χ_t is a log-normal random variable described by its two first centered moments $\mathrm{M}_t = \mathbb{E}[\ln(\chi_t)]$ and $\mathrm{V}_t = \mathrm{Var}[\ln \chi_t]$ that can easily be computed:

$$\mathrm{M}_t = \ln\left(\frac{A_0}{P(0,t)}\right) + \int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t)+\rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2} - r_g\right) \mathrm{d}u$$

and

$$\mathrm{V}_t = \int_0^t (\sigma^2 + \sigma_P^2(u,t) + 2\rho\sigma\sigma_P(u,t))\,\mathrm{d}u$$

Let us give the moments of $\ln(\chi_t)$ for an exponential volatility structure

$$\mathrm{M}_t = \ln\left(\frac{A_0}{P(0,t)}\right) + \frac{\nu^2}{4a^3} - \left(\frac{\nu^2}{2a^2} + \frac{\rho\sigma\nu}{a} + \frac{\sigma^2}{2} + r_g\right) t - \frac{\nu^2}{4a^3}\,\mathrm{e}^{-2at}\left(\frac{\nu^2}{2a^3} + \frac{\rho\sigma\nu}{a^2}\right)\mathrm{e}^{-a(T-t)}$$
$$- \left(\frac{\nu^2}{a^3} + \frac{\rho\sigma\nu}{a^2}\right)\mathrm{e}^{-aT} + \frac{\nu^2}{2a^3}\,\mathrm{e}^{-a(T+t)},$$
$$\mathrm{V}_t = 2\nu\frac{\nu + a\rho\sigma}{a^3}\,\mathrm{e}^{-at} - \frac{\nu^2}{2a^3}\,\mathrm{e}^{-2at} - \frac{3\nu^2}{2a^3} - \frac{2\rho\sigma\nu}{a^2} + \left(\sigma^2 + \frac{2\rho\sigma\nu}{a} + \frac{\nu^2}{a^2}\right) t$$

We need to compute the covariance between $\ln(\chi_t)$ and $\ln(\chi_s)$:

$$\mathrm{C}(s,t) = \int_0^{s\wedge t} (\sigma^2 + \rho\sigma(\sigma_P(u,t) + \sigma_P(u,s)) + \sigma_P(u,s)\sigma_P(u,t))\,\mathrm{d}u$$

In the case of the Hull and White volatility, we obtain (with $s < t$):

$$\mathrm{C}(s,t) = -\left(\frac{\rho\sigma\nu}{a^2} + \frac{\nu^2}{a^3}\right) + \left(\sigma^2 + \frac{2\rho\sigma\nu}{a} + \frac{\nu^2}{a^2}\right) s + \left(\frac{\rho\sigma\nu}{a^2} + \frac{\nu^2}{a^3}\right)\mathrm{e}^{-as} + \left(\frac{\rho\sigma\nu}{a^2} + \frac{\nu^2}{a^3}\right)\mathrm{e}^{-at}$$
$$- \left(\frac{\rho\sigma\nu}{a^2} + \frac{\nu^2}{2a^3}\right)\mathrm{e}^{-a(t-s)} - \frac{\nu^2}{2a^3}\,\mathrm{e}^{-a(t+s)}$$

Besides, the conditional law of $\ln(\chi_t)$ given $\ln(\chi_s)$ is Gaussian with mean $\hat{m}(s,t)$ and variance $\hat{V}(s,t)$. The conditional moments of $\ln(\chi_t)$ are

$$\hat{m}(s,t) = \mathrm{M}_t + \frac{\mathrm{C}(s,t)}{\mathrm{V}_s}(\ln(\chi_s) - \mathrm{M}_s), \qquad \hat{V}(s,t) = \mathrm{V}_t - \frac{\mathrm{C}(s,t)^2}{\mathrm{V}_s} \tag{19}$$

A.2. Moments of the processes r_t and l_t

We work under the forward-neutral measure. The instantaneous interest rate r is an Ornstein–Uhlenbeck process. We compute its moments and those of l associated with the assets process. Define B_a by:

$$B_a(u) = \frac{1}{a}(1 - \mathrm{e}^{-au})$$

r is a Gaussian process, therefore it is possible (after integrating (9)) to compute its two centered conditional moments with respect to the tribe $\mathcal{F}$ generated by r:

$$\mathbb{E}[r_t|\mathcal{F}_u] = \mathrm{e}^{-a(t-u)}\, r_u + \left(\theta a - \frac{\nu^2}{a}\right) B_a(t-u) + \frac{\nu^2}{a}\,\mathrm{e}^{-a(T-t)}\, B_{2a}(t-u)$$

and

$$\mathrm{Var}[r_t|\mathcal{F}_u] = \nu^2 B_{2a}(t-u)$$

and for $s < t$

$$\mathrm{Cov}(r_s, r_t|\mathcal{F}_u) = \frac{\nu^2}{2a}\,\mathrm{e}^{-a(s+t)}(\mathrm{e}^{2as} - \mathrm{e}^{2au}) = \nu^2\,\mathrm{e}^{-a(t-s)}\, B_{2a}(s-u)$$

Let us now examine the moments of the process $l_t = \ln(\chi_t) = \ln(A_t) - r_g t$ obeying the SDE

$$\mathrm{d}l_t = \left(r_t - r_g - \frac{\sigma^2}{2} - \sigma\rho\nu B_a(T-t)\right)\mathrm{d}t + \sigma\rho\,\mathrm{d}Z_1^{\mathrm{Q}_T} + \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{Q}_T} \tag{20}$$

where $Z_1^{\mathrm{Q}_T}$ and $Z_2^{\mathrm{Q}_T}$ are two independent Brownian under the T-forward neutral measure.

We integrate l_t; it can be expressed in terms of r_t, $Z_1^{\mathrm{Q}_T}$ and $Z_2^{\mathrm{Q}_T}$. l is a Gaussian process. After some computations we obtain:

$$\begin{aligned}\mathbb{E}[l_t|\mathcal{F}_u] &= l_u - \left(r_g + \frac{\sigma^2}{2} + \frac{\sigma\rho\nu}{a} - \theta + \frac{\nu^2}{a^2}\right)(t-u) - \frac{\nu^2}{a^2}\,\mathrm{e}^{-a(T-t)} B_{2a}(t-u)\\ &\quad + \left(r_u - \theta + \frac{\nu^2}{a^2} + \frac{\nu^2}{a^2}\,\mathrm{e}^{-a(T-t)} + \frac{\sigma\rho\nu}{a}\,\mathrm{e}^{-a(T-t)}\right) B_a(t-u),\\ \mathrm{Var}[l_t|\mathcal{F}_u] &= \left(\sigma^2 + \frac{\nu^2}{a^2} + 2\frac{\sigma\rho\nu}{a}\right)(t-u) - 2\left(\frac{\nu^2}{a^2} + \frac{\sigma\rho\nu}{a}\right) B_a(t-u) + \frac{\nu^2}{a^2} B_{2a}(t-u).\end{aligned}$$

If $s < t$:

$$\begin{aligned}\mathrm{Cov}(l_s, l_t|\mathcal{F}_u) &= \frac{\nu^2}{a^2}\,\mathrm{e}^{-a(t-s)} B_{2a}(s-u) + \left(\sigma^2 + \frac{2\sigma\rho\nu}{a} + \frac{\nu^2}{a^2}\right)(s-u)\\ &\quad - \left(\frac{\nu^2}{a^2} + \frac{\sigma\rho\nu}{a}\right)(\mathrm{e}^{-a(t-s)} + 1) B_a(s-u)\end{aligned}$$

A.3. Covariances between l_t and r_t

The processes l_t and r_t are correlated through $Z_1^{\mathrm{Q}_T}$ and have the conditional covariance

$$\mathrm{Cov}(l_t, r_t|\mathcal{F}_u) = -\frac{\nu^2}{a} B_{2a}(t-u) + \left(\frac{\nu^2}{a} + \rho\sigma\nu\right) B_a(t-u)$$

Besides, we need

$$\mu(r_t, l_s, r_s) = \mathbb{E}[l_t|\mathcal{F}_s] + \frac{\mathrm{Cov}(l_t, r_t|\mathcal{F}_s)}{\mathrm{Var}[r_t|\mathcal{F}_s]}(r_t - \mathbb{E}[r_t|\mathcal{F}_s]), \qquad \Sigma^2(r_t, l_s, r_s) = \mathrm{Var}[l_t|\mathcal{F}_s] - \frac{\mathrm{Cov}(l_t, r_t|\mathcal{F}_s)^2}{\mathrm{Var}[r_t|\mathcal{F}_s]}$$

Finally, we also need the following results to compute E_6:

$$\mathbb{E}\left[\int_u^T r_s\,\mathrm{d}s|\mathcal{F}_u\right] = (r_u-\theta)B_a(T-u)+\frac{\nu^2}{a}B_a(T-u)^2+\frac{\nu^2}{a^2}\,\mathrm{e}^{-au}B_{2a}(T-u)+\left(\theta-\frac{\nu^2}{a^2}\right)(T-u)$$

and

$$\mathrm{Var}\left[\int_u^T r_s\,\mathrm{d}s|\mathcal{F}_u\right] = \frac{\nu^2}{a^2}(B_{2a}(T-u)+T-u-2B_a(T-u))$$

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