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Pricing derivatives with barriers in a stochastic interest rate environment

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Abstract

This paper develops a general valuation approach to price barrier options when the term structure of interest rates is stochastic. These products' barriers may be constant or stochastic, in particular we examine the case of discounted barriers (at the instantaneous interest rate). So, in practice, we extend Rubinstein and Reiner [1991. Breaking down the barriers. Risk 4(8), 28–35], who give closed-form formulas for pricing barrier options in a Black and Scholes context, to the case of a Vasicek modeling of interest rates. We are therefore in the situation of pricing barrier options semi-explicitly or explicitly (depending on the shape of the barrier) with stochastic Vasicek interest rates. The model is illustrated with a specific contract, an up and out call with rebate, hence a typical barrier option. This example is merely here to show how any standard barrier option can be priced and its Greeks be obtained in such a context. The validity of the approximation is analyzed and the sensitivity to the barrier level and to discretization schemes are also derived.

JEL classification: C60; G10

Keywords: Change of Numéraire; Vasicek model; Barrier option; Markovian approximation

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1. Introduction

In this article, we focus our analysis on the pricing of financial contracts with barriers in a stochastic interest rate environment. The applications of barrier options are multiple and go far beyond the study of derivative products. Barrier options are building blocks of diverse fields such as investment choice theory, the study of the capital structure of the firm (see the standard reference of Black and Cox, 1976 for instance, or the interesting contribution of François and Morellec, 2004), or life insurance (see for instance Grosen and Jørgensen, 2002). Recall that these contracts payoffs depend on whether or not the price of their underlying assets cross a barrier from above or from below. They are the essential part of the standard structured products that are guaranteeing the maximum of a capital and the performance of a financial index.

Barrier options have been studied in great detail for a long time. Under the assumption of a unique and constant interest rate, closed-form solutions were given by Merton (1974) for down and out calls, then by Rubinstein and Reiner (1991) for vanilla barrier options. Other contributions include the works of Geman and Yor (1996) and Pelsser (2000) who priced double barrier options, and the innovative article of Chesney et al. (1997) who introduced Parisian barrier options. The payoff of the latter contracts depends on the time spent above or below the barrier. Later on, Linetsky (1999) pioneered step options. In all these papers, the standard Black and Scholes framework is the starting point and in particular the risk-free interest rate is assumed constant. For short term contracts, a constant term structure of interest rates can be considered reasonable; yet, for medium or long term notes this assumption cannot hold.

Lots of structured products currently traded on the American Stock Exchange involve barrier options and some of them are long term products. Indeed 25.7% of equity-linked notes[1] have their payments driven by a triggered event based on the trajectory of the underlying stock. These equity-linked securities with embedded barrier options represent a total volume of $1,109,518,000. The three bigger issuers in the US are: Wachovia Corporation (Enhanced Yield Securities), Morgan Stanley (HITS: High Income Trigger Securities), Citigroup Funding Inc. (EKLS: Equity Linked Securities). The maturities of these products can be very long in the real market. A lot of barrier equity-linked securities (such as described above) are 1-year contracts. Yet, a reasonable amount of long term barrier index linked notes is currently traded on the American Stock Exchange. Typical medium-index barrier index linked notes are issued by Lehman Brothers, and are linked to the S&P 500 Index, Dow Jones STOXX 50 Index, or Russell 2000 Index (their maturities are ranging between 4 and 5 years). Medium-term notes are also existing and popular products in Europe: we can take for example the famous equity winners, or twin wins, which are basically linear combinations of in and out barrier options with a maturity of 5 years. For these products, it is fully relevant to take into account the

[1] Source: www.amex.com, November 2006.

stochasticity of interest rates. More on structured products can also be found in Wystup (2006).

The study of exotic barrier options in the context of stochastic interest rates is a rather difficult problem. It is usually solved in the financial industry by means of Monte-Carlo simulations or partial differential equations. This article takes into account a stochastic term structure of interest rates to price barrier options by means of closed or semi-closed form formulas in continuous time. When a particular type of barrier is chosen, as in Briys and de Varenne (1997) or Kraft (2004), closed form formulas can be obtained. Here we suggest a general methodology adapted from Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001). To do so, our framework considers a type of Markovian approximation due to Fortet (1943) and used by Longstaff and Schwartz (1995) to value risky debt. Collin-Dufresne and Goldstein (2001) generalized Fortet's approximation to the case of two-dimensional Markov processes. We use their extension to price exotic barrier options. In the actuarial field, Bernard et al. (2005) priced successfully life insurance contracts owning many covenants in a similar stochastic context. Our paper goes beyond this article to show how the extended Fortet method can be used in the finance realm for barrier options. Moreover we show how the sensitivities of barrier options can easily be obtained since the convergence of our method is much smoother than classical numerical methods (Monte-Carlo simulations, trees algorithms, numerical schemes for partial differential equations).

This article is organized as follows. In the first section, we show how standard barrier options can be priced with semi-closed-form formulas, when the interest rate process is stochastic and of the Vasicek type. This section is therefore the direct extension of the work of Rubinstein and Reiner (1991) to a setting with stochastic interest rates. The second section illustrates our approach with a particular exotic contract, the shark option (a typical example of medium-term note recently issued in Europe), which is in fact a barrier option with rebate. This shark option is used for the sake of illustration; we could have of course chosen an other medium-term product for the same purposes. This section also develops a subsetting where it is possible to reduce the semi-closed form formulas to closed-form formulas, while keeping the randomness of the underlying interest rate process. The last section applies our results in the context of a numerical analysis.

2. Standard barrier options in a Vasicek model

Let us start by considering a financial market with a primary asset, say a stock S, on which a barrier option is written. The underlying asset price is assumed to follow a geometric Brownian motion. The interest rate model is a Vasicek one, in particular the instantaneous interest rate r enjoys the Markovian property. The uncertainty is modeled by a filtered space $(\Omega, F, \{F_t\}_{t\geqslant 0}, \Pi)$ where Ω is the usual fundamental space, $\{F_t\}_{t\geqslant 0}$ is the filtration generated by the Brownian motions, and Π is the historical probability measure. Trading takes place continuously and the prices of all assets follow correlated diffusions. In particular, the interest rate process is correlated to the stock process, or put differently, the economy is driven by two correlated Brownian motions. The market is complete and frictionless, and Q denotes the risk-neutral probability.

Because standard barrier options can be *up* or *down*, *in* or *out*, *call* or *put* options, there are eight types of such options. For the sake of brevity, we will only price in this section call options (the put option formulas can be obtained straightforwardly from parity relationships), that is to say *up* and *out*, *up* and *in*, *down* and *out*, and *down* and *in* barrier call options.

Denoting by T the maturity of the options, by K their strike, and by H their barrier level, one can write the following arbitrage-free pricing formulas for the *up* and *out*, and *up* and *in* call options:

$$\begin{cases} C^{\mathrm{uo}} = E_Q\left(\mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}(S_T - K)^+ \mathbb{1}_{S_{\max} \leqslant H}\right), \\ C^{\mathrm{ui}} = E_Q\left(\mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}(S_T - K)^+ \mathbb{1}_{S_{\max} > H}\right). \end{cases} \tag{1}$$

As concerns the *down* and *out*, and *down* and *in* calls, they admit the following valuation formulas:

$$\begin{cases} C^{\mathrm{do}} = E_Q\left(\mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}(S_T - K)^+ \mathbb{1}_{S_{\min} \geqslant H}\right), \\ C^{\mathrm{di}} = E_Q\left(\mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}(S_T - K)^+ \mathbb{1}_{S_{\min} < H}\right). \end{cases} \tag{2}$$

The goal of this section will be to show how the formulas in (1) and (2) can be priced in semi-closed form.

2.1. Pricing framework

We will need in the coming developments to use the forward-neutral dynamics of the stock, of the default-free zero-coupons and of the stock expressed in units of default-free zero-coupon. The dynamics of the default-free zero-coupons $P(t,T)$ classically write, in the historical world, as

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \lambda(t,T)\,\mathrm{d}t - \sigma_P(t,T)\,\mathrm{d}Z_1(t),$$

where $\lambda(t,T)$ is their expected return, $\sigma_P(t,T)$ their volatility, and Z_1 a standard Brownian motion under Π. In this article, we assume that $\sigma_P(t,T)$ is deterministic.

The option's underlying price at time t, denoted by S_t, is modeled by a geometric Brownian motion:

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}Z_2(t),$$

where Z_2 is a standard Brownian motion correlated with Z_1: we define the correlation coefficient ρ by $\mathrm{d}Z_1\,\mathrm{d}Z_2 = \rho\,\mathrm{d}t$. The instantaneous expected return μ

could be any square-integrable adapted process. This will not intervene in the following because we are only interested in pricing under no arbitrage in this paper.

These dynamics are given in the historical universe. Using standard results from risk-neutral analysis, we know that there exists a unique probability measure Q under which the discounted price of securities are martingales. After decorrelating the above Brownian motions, we can write under Q:

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r_t\,\mathrm{d}t - \sigma_P(t,T)\,\mathrm{d}\widehat{Z}_1(t)$$

which is standard in this context, see Björk (2004), and for the underlying's price:

$$\frac{\mathrm{d}S_t}{S_t} = r_t\,\mathrm{d}t + \sigma\Big(\rho\,\mathrm{d}\widehat{Z}_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}\widehat{Z}_2(t)\Big),$$

where $\widehat{Z}_1$ and $\widehat{Z}_2$ are now two uncorrelated Q-Brownian motions.

Using Itō's lemma, we can express the risk-neutral dynamics of S_t and $P(t,T)$ as

$$S_t = S_0 \exp\left(\int_0^t r_u\,\mathrm{d}u - \frac{1}{2}\sigma^2 t + \int_0^t \rho\sigma\,\mathrm{d}\widehat{Z}_1(u) + \int_0^t \sigma\sqrt{1-\rho^2}\,\mathrm{d}\widehat{Z}_2(u)\right) \tag{3}$$

and

$$P(t,T) = P(0,T)\exp\left(\int_0^t r_u\,\mathrm{d}u - \frac{1}{2}\int_0^t \sigma_P^2(u,T)\,\mathrm{d}u - \int_0^t \sigma_P(u,T)\,\mathrm{d}\widehat{Z}_1(u)\right). \tag{4}$$

We now aim at writing the dynamics of S in the T-forward-neutral universe. This universe is associated with the couple $(Q_T, P(t,T))$ (i.e. measure, numéraire). First, we start using the martingale property of the relative price $S_t/P(t,T)$, which reads

$$\frac{S_t}{P(t,T)} = \frac{S_0}{P(0,T)}\exp\left(\begin{array}{c}\displaystyle\int_0^t (\sigma_P(u,T)+\rho\sigma)\,\mathrm{d}Z_1^{\mathrm{T}}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{T}}(u) \\ \displaystyle -\frac{1}{2}\int_0^t ((\sigma_P(u,T)+\rho)^2 + \sigma^2(1-\rho^2))\,\mathrm{d}u\end{array}\right) \tag{5}$$

and we readily set

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left(\begin{array}{c}\displaystyle -\int_0^t (\sigma_P(u,T)-\sigma_P(u,t))\,\mathrm{d}Z_1^{\mathrm{T}}(u) \\ \displaystyle +\frac{1}{2}\int_0^t (\sigma_P(u,T)-\sigma_P(u,t))^2\,\mathrm{d}u\end{array}\right),$$

where Z_1^{T} and Z_2^{T} are two uncorrelated Q_T-Brownian motions, defined by the two following relationships: $\mathrm{d}Z_1^{\mathrm{T}}(t) = \mathrm{d}\widehat{Z}_1(t) + \sigma_P(t,T)\,\mathrm{d}t$ and $\mathrm{d}Z_2^{\mathrm{T}}(t) = \mathrm{d}\widehat{Z}_2(t)$.

Finally, we can obtain the forward-neutral expression of S_t that is going to be used in the remainder of this paper:

$$S_t = \frac{S_0}{P(0,t)} \exp\left(\begin{array}{c} \displaystyle\int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t)+\rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2} \right) \mathrm{d}u \\ \displaystyle + \int_0^t (\sigma_P(u,t)+\rho\sigma)\,\mathrm{d}Z_1^{\mathrm{T}}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{T}}(u) \end{array} \right) \tag{6}$$

or equivalently

$$\ln(S_t) = \ln\left(\frac{S_0}{P(0,t)}\right) + \left(\begin{array}{c} \displaystyle\int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t)+\rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2} \right) \mathrm{d}u \\ \displaystyle + \int_0^t (\sigma_P(u,t)+\rho\sigma)\,\mathrm{d}Z_1^{\mathrm{T}}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{T}}(u) \end{array} \right).$$

Hence, under Q_T, the underlying price is lognormal, and $\ln(S)$ is a Gaussian process. Denoting it by l, we can also remark that

$$\mathrm{d}l_t = \left(r_t - \frac{\sigma^2}{2} - \sigma\rho\sigma_P(t,T) \right) \mathrm{d}t + \sigma\rho\,\mathrm{d}Z_1^{\mathrm{T}}(t) + \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{T}}(t). \tag{7}$$

We will also need the following moments: M_t, V_t and $Cov(v,t)$, $v \leqslant t$, which, respectively, denote the mean, variance and auto-covariance of the underlying. Their generic expressions are

$$\begin{cases} M_t = \ln\left(\dfrac{S_0}{P(0,t)}\right) + \displaystyle\int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t)+\rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2} \right) \mathrm{d}u, \\ V_t = \displaystyle\int_0^t (\sigma^2 + \sigma_P^2(u,t) + 2\rho\sigma\sigma_P(u,t))\,\mathrm{d}u, \\ Cov(v,t) = \displaystyle\int_0^v (\sigma^2 + \rho\sigma(\sigma_P(u,t)+\sigma_P(u,v)) + \sigma_P(u,v)\sigma_P(u,t))\,\mathrm{d}u. \end{cases} \tag{8}$$

Furthermore, using standard probabilistic results on bidimensional Gaussian vectors, we know that the conditional law of $\ln(S_t)$ given $(\ln(S_v) = \ln(H))$, where $\ln(H) = h$ is an arbitrary given level, is normal and possesses the following mean $\widehat{M}$ and variance $\widehat{V}$:

$$\begin{cases} \widehat{M}(v,t) = M_t + \dfrac{Cov(v,t)}{V_v}(\ln(H) - M_v), \\ \widehat{V}(v,t) = V_t - \dfrac{Cov^2(v,t)}{V_v}. \end{cases}$$

Standard computations enable computing explicitly the above moments in the two cases of linear and exponential volatility structures. The results for M, V and Cov are given in Appendix B (from them, one obtains straightforwardly the expressions for $\widehat{M}$ and $\widehat{V}$) in the case of an exponential structure of volatility which corresponds to the Vasicek model.

2.2. Semi-closed form formulas

We can now start deriving the quasi-closed expressions of the arbitrage-free formulas (1) and (2) of barrier call options. We start with the *up* and *in* and the *up* and *out* options.

2.2.1. Pricing up call options

To begin with, we can reexpress the formula of the *up* and *in* option in (1) in the forward-neutral universe:

$$C^{\mathrm{ui}} = P(0,T)E_{Q_T}((S_T-K)^+\mathbb{1}_{S_{\max}>H}).$$

This can alternatively be written as

$$\frac{C^{\mathrm{ui}}}{P(0,T)} = E_{Q_T}(S_T\mathbb{1}_{S_T>K}\mathbb{1}_{S_{\max}>H} - K\mathbb{1}_{S_T>K}\mathbb{1}_{S_{\max}>H})$$

or as

$$\frac{C^{\mathrm{ui}}}{P(0,T)} = E_{Q_T}(S_T\mathbb{1}_{S_T>K}\mathbb{1}_{S_{\max}>H}) - KQ_T(S_T>K, S_{\max}>H).$$

Finally, the *up* and *in* call option price C^{ui} is given by

$$\frac{C^{\mathrm{ui}}}{P(0,T)} = \mathscr{A} - K\mathscr{B}, \tag{9}$$

where

$$\begin{cases} \mathscr{A} = E_{Q_T}(S_T\mathbb{1}_{S_T>K}\mathbb{1}_{S_{\max}>H}), \\ \mathscr{B} = Q_T(S_T>K, S_{\max}>H). \end{cases}$$

Note the following problem that appears in the computation of $\mathscr{A}$ and $\mathscr{B}$: the explicit expression of the law of $S_{\max}$, and *a fortiori* of the joint law of $S_{\max}$ and S_T, is not known. The event $\{S_{\max}>H\}$ is indeed equivalent to the first passage time of the process S through the barrier level H occurring before the maturity T of the option. Let us denote by γ^{u} this first passage time ('u' for an 'up' barrier). One readily has $\{S_{\max}>H\} = \{\gamma^{\mathrm{u}} \leqslant T\}$. We do not know the explicit joint distribution of γ^{u} and $r_{\gamma^{\mathrm{u}}}$; yet, a discretized version of it can be obtained using the recursive argument of Collin-Dufresne and Goldstein (2001). In Appendix A we expose this algorithm, titled the extended Fortet method, along a new and clean presentation (relying in particular on distributions and not on densities).

The distribution function of the random vector $(r_{\gamma^{\mathrm{u}}}, \gamma^{\mathrm{u}})$ at time t under the T-forward-neutral measure Q_T is unknown, as previously said. We approximate it by discretizing along the time and interest rate dimensions. The interval $[0,T]$ is subdivided into n_T subperiods of length $\delta_t = T/n_T$, and the interest rate is subdivided between $r_{\min}$ and $r_{\max}$ into n_r intervals of length $\delta_r = (r_{\max}-r_{\min})/n_r$. Finally, we denote by $t_j = j\delta_t$ and $r_i = r_{\min} + i\delta_r$ the discretized values of time and interest rate. Next, denote also by

$$q^{\mathrm{u}}(i,j) = Q_T(r_{\gamma^{\mathrm{u}}} \in [r_i, r_{i+1}], \gamma^{\mathrm{u}} \in [t_j, t_{j+1}])$$

the discretized version of the first-passage time distribution. We show below that semi-closed-form formulas for $\mathscr{A}$ and $\mathscr{B}$ are written as

$$\begin{cases} \mathscr{A} \approx \displaystyle\sum_{j=0}^{n_T}\sum_{i=0}^{n_r} \kappa(\widehat{\mu}_{t_j,T}; \widehat{\Sigma}_{t_j,T}; K) q^{\mathrm{u}}(i,j), \\ \mathscr{B} \approx \displaystyle\sum_{j=0}^{n_T}\sum_{i=0}^{n_r} \mathscr{N}\left(\frac{\ln(K) - \widehat{\mu}_{t_j,T}}{\sqrt{\widehat{\Sigma}^2_{t_j,T}}}\right) q^{\mathrm{u}}(i,j), \end{cases} \tag{10}$$

where κ is defined, for a Gaussian random variable X following the law $\mathscr{N}(m, \sigma^2)$, by

$$\kappa(m; \sigma; a) = E(\mathrm{e}^X \mathbb{1}_{\mathrm{e}^X > a}) = \exp\left(m + \frac{\sigma^2}{2}\right) \mathscr{N}\left(\frac{m + \sigma^2 - \ln(a)}{\sigma}\right)$$

and $\widehat{\mu}_{s,T}$ and $\widehat{\Sigma}_{s,T}$ are the two first centered moments of l_T conditional on $\mathscr{F}_s$, namely

$$\begin{cases} \widehat{\mu}_{s,T} = E_{Q_T}[l_t \mid \mathscr{F}_s], \\ \widehat{\Sigma}_{s,T} = Var_{Q_T}[l_t \mid \mathscr{F}_s] \end{cases}$$

whose detailed expressions can be found in Appendices A and B.

The above development of $\mathscr{A}$ can be justified as follows. Start with the expression

$$\mathscr{A} = E_{Q_T}[S_T \mathbb{1}_{\ln(S_T) > \ln(K)} \mathbb{1}_{\gamma^{\mathrm{u}} \leqslant T}]$$

which can be simplified according as

$$\mathscr{A} = E_{Q_T}[\mathrm{e}^{l_T} \mathbb{1}_{l_T > \ln(K)} \mathbb{1}_{\gamma^{\mathrm{u}} \leqslant T}].$$

Using the conditional distribution of l_T on the information $\mathscr{F}_s$, we can write

$$\mathscr{A} = \int_0^T \int_{-\infty}^{+\infty} E_{Q_T}[\mathrm{e}^{l_T} \mathbb{1}_{l_T > \ln(K)} \mid r_{\gamma^{\mathrm{u}}} = r, \gamma^{\mathrm{u}} = s] Q_T(r_{\gamma^{\mathrm{u}}} \in \mathrm{d}r, \gamma^{\mathrm{u}} \in \mathrm{d}s).$$

Given $\widehat{\mu}_{s,T}$ and $\widehat{\Sigma}_{s,T}$ the two first centered moments of l_T conditional on $\mathscr{F}_s$ and defining κ by $\kappa(m; \sigma; a) = E(\mathrm{e}^X \mathbb{1}_{\mathrm{e}^X > a})$, one can readily write

$$\mathscr{A} = \int_0^T \int_{-\infty}^{+\infty} \kappa(\widehat{\mu}_{s,T}; \widehat{\Sigma}_{s,T}; K) Q_T(r_{\gamma^{\mathrm{u}}} \in \mathrm{d}r, \gamma^{\mathrm{u}} \in \mathrm{d}s).$$

Finally, one obtains the following discretized approximation[2] of $\mathscr{A}$:

$$\mathscr{A} \approx \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} \kappa(\widehat{\mu}_{t_j,T}; \widehat{\Sigma}_{t_j,T}; K) q^{\mathrm{u}}(i,j).$$

[2]Because the integrated functions are smooth enough, a first order maximization of the discretization can be obtained by considering that the volume measured the double integral is a sum of small volumes, each of them comprised between two parallelepipeds. This yields therefore the following first order maximization of the discretization error: $\varepsilon = \sum_{j=0}^{n_T}\sum_{i=0}^{n_r} |\kappa(j)q^{\mathrm{u}}(i,j) - \kappa(j-1)q^{\mathrm{u}}(i-1,j-1)|$ where we remark that $\widehat{\mu}_{s,T}$ and $\widehat{\Sigma}_{s,T}$ depend on l_s and r_s, so where $\kappa(j)$ is a simplified notation for $\kappa(\widehat{\mu}_{t_j,T}; \widehat{\Sigma}_{t_j,T}; K)$.

This explains the discretized approximation of $\mathscr{A}$ in formula (10). The one for $\mathscr{B}$ can be obtained in a similar way. Start considering

$$\mathscr{B} = E_{Q_T}[\mathbb{1}_{l_T > \ln(K)} \mathbb{1}_{\gamma^{\mathrm{u}} \leqslant T}]$$

which can be developed as

$$\mathscr{B} = \int_0^T \int_{-\infty}^{+\infty} E_{Q_T}[\mathbb{1}_{l_T > \ln(K)} \,|\, r_{\gamma^{\mathrm{u}}} = r, \gamma^{\mathrm{u}} = s] Q_T(r_{\gamma^{\mathrm{u}}} \in \mathrm{d}r, \gamma^{\mathrm{u}} \in \mathrm{d}s)$$

or as

$$\mathscr{B} = \int_0^T \int_{-\infty}^{+\infty} Q_T[l_T > \ln(K) \,|\, r_{\gamma^{\mathrm{u}}} = r, \gamma^{\mathrm{u}} = s] Q_T(r_{\gamma^{\mathrm{u}}} \in \mathrm{d}r, \gamma^{\mathrm{u}} \in \mathrm{d}s).$$

This is equivalent to

$$\mathscr{B} = \int_0^T \int_{-\infty}^{+\infty} \mathscr{N}\left(\frac{\ln(K) - \widehat{\mu}_{t_j,T}}{\sqrt{\widehat{\Sigma}^2_{t_j,T}}}\right) Q_T(r_{\gamma^{\mathrm{u}}} \in \mathrm{d}r, \gamma^{\mathrm{u}} \in \mathrm{d}s)$$

and, finally, one obtains

$$\mathscr{B} \approx \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} \mathscr{N}\left(\frac{\ln(K) - \widehat{\mu}_{t_j,T}}{\sqrt{\widehat{\Sigma}^2_{t_j,T}}}\right) q^{\mathrm{u}}(i,j).$$

Therefore, one has all the necessary elements to compute the *up* and *in* barrier call options formula (9). As mentioned above, the terms $q^{\mathrm{u}}(i,j)$ can be computed using the methodology in Appendix A.

Now, to price an *up* and *out* call, it is sufficient to use the following parity relationship:

$$C^{\mathrm{uo}} = P(0,T) E_{Q_T}((S_T - K)^+) - C^{\mathrm{ui}}$$

noting that

$$P(0,T) E_{Q_T}((S_T - K)^+) = P(0,T)\left(\kappa(M_T, S_T, K) - K\mathscr{N}\left(\frac{\ln(K) - M_T}{\sqrt{V_T}}\right)\right), \tag{11}$$

where all the moments and symbols are the same as defined before. Let us now come to the pricing of *down* barrier call options.

2.2.2. Pricing down call options

We will sketch the main ideas and formulas in this paragraph; clearly, all the derivations are analogical to the ones of the previous paragraphs. We start with the pricing of *down* and *in* call options. Their valuation formula in (2) can be reexpressed in the forward-neutral universe as

$$C^{\mathrm{di}} = P(0,T) E_{Q_T}((S_T - K)^+ \mathbb{1}_{S_{\min} < H}).$$

Next, we denote by γ^{d} the first passage time by S of a *down* barrier H. Defining $S_{\min}$ on $[0, T]$, one has: $\{S_{\min} < H\} = \{\gamma^{\mathrm{d}} \leqslant T\}$. By analogy with (9), we write

$$\frac{C^{\mathrm{di}}}{P(0,T)} = \mathscr{C} - K\mathscr{D}, \tag{12}$$

where

$$\begin{cases} \mathscr{C} = E_{Q_T}(S_T \mathbb{1}_{S_T > K} \mathbb{1}_{\gamma^{\mathrm{d}} \leqslant T}), \\ \mathscr{D} = Q_T(S_T > K, \gamma^{\mathrm{d}} \leqslant T) \end{cases}$$

and where, by analogy with the previous developments, one has the approximations

$$\begin{cases} \mathscr{C} = \displaystyle\sum_{j=0}^{n_T} \sum_{i=0}^{n_r} \kappa(\widehat{\mu}_{t_j,T}; \widehat{\Sigma}_{t_j,T}; K) q^{\mathrm{d}}(i,j), \\ \mathscr{D} = \displaystyle\sum_{j=0}^{n_T} \sum_{i=0}^{n_r} \mathscr{N}\left(\frac{\ln(K) - \widehat{\mu}_{t_j,T}}{\sqrt{\widehat{\Sigma}^2_{t_j,T}}}\right) q^{\mathrm{d}}(i,j). \end{cases} \tag{13}$$

As concerns the *down* and *out* call, its pricing follows readily from the following parity relationship:

$$C^{\mathrm{do}} = P(0,T) E_{Q_T}((S_T - K)^+) - C^{\mathrm{di}},$$

where the first term is given by Eq. (11).

To conclude this section, we have constructed semi-closed-form expressions for standard barrier options under a Vasicek model for the interest rate dynamics. The term 'semi' in 'semi-closed form' refers to the fact that the $q^{\mathrm{u}}(i,j)$ and $q^{\mathrm{d}}(i,j)$ factors are only approximations of the first passage time distribution. In practice, and as the final section shows, these semi-closed form formulas can be computed extremely quickly. The next section shows how this methodology can be used to price a particular exotic contract.

3. Pricing a structured barrier option

Our aim will now be to shed some light on the use of the above method to price some exotic contracts. We start defining the 'shark' option, which was introduced a couple of years ago by the Equity desk of an international bank.

3.1. The shark index option

In its most basic form, a shark option is an option whose holder is entitled to receive a rebate at expiry if the underlying index hits a barrier and a European payoff otherwise. The latter depends on the value of the underlying index at expiry and may take the form of a European call or a functional of it, as the following developments will show. The underlying index may be a financial asset, an interest rate, an exchange rate or an Equity Index. In full generality, it is correlated to the interest rates. Here, we assume that payments are always settled at expiry (ranging typically from 1 to 5 years for these options). The presence of a barrier decreases the premium, compared to vanilla options. The barrier can be hit from below or from above and can be a knock-in or a knock-out one. It may also be constant, deterministic or stochastic.

For the sake of clarity, we shall consider from now on a special kind of shark option. Yet, the reader should keep in mind that our method can be applied to value many other similar products. Let us describe more precisely our contract: it is a medium-term structured note, having a 1–5 year maturity, guaranteeing the investor (purchaser of the shark) $\beta\%$ of his capital, and linearly linked to an Equity Index. However, this link is cut as soon as the growth rate of the index is equal or greater than $\alpha\%$ during the shark's life, in which case the investor receives $\beta\%$ of his initial investment at the end. In formal terms, the investor receives at expiry time T:

$$\begin{cases} M(1+R_T) & \text{if } S_{\max} \leqslant (1+\alpha)S_0, \\ M\beta & \text{otherwise}, \end{cases} \tag{14}$$

where $R_T = (S_T - S_0)^+/S_0$, M is a notional amount, S_t is the index at time t or the underlying shark's price at time t, and $S_{\max}$ is the maximum of S before the shark's maturity, that is, over $[0, T]$.

As concerns α and β, they, respectively, describe the barrier level and the value of the rebate, and we let them satisfy $\alpha > 0$ and $0 < \beta < 1 + \alpha$. We call this structured product a standard shark and, without loss of generality, we assume $M = 1$ for the sake of simplicity. We give a numerical example in Section 4. We denote by H the barrier level:

$$H = (1+\alpha)S_0.$$

The payoff at maturity (assuming $M = 1$) then writes

$$(1+R_T)\mathbb{1}_{S_{\max} \leqslant H} + \beta \mathbb{1}_{S_{\max} > H}. \tag{15}$$

In fact, one has $1 + R_T = 1 + (S_T - S_0)^+/S_0$. This allows rewriting the payoff as

$$1 + \frac{1}{S_0}(S_T - S_0)^+ \mathbb{1}_{S_{\max} \leqslant H} + (\beta - 1)\mathbb{1}_{S_{\max} > H}. \tag{16}$$

Technically, a shark option is merely an *up* and *out* barrier call option with a rebate. Indeed, $(S_T - S_0)^+ \mathbb{1}_{S_{\max} \leqslant H}$ is the payoff of an *up* and *out* call on the underlying S, with a strike price $K = S_0$, and a barrier H. Due to the presence of a barrier condition, the payoff at maturity is discontinuous. Moreover we call 'shark option' this contract because of its shape (see Fig. 1). Recent studies show that such discontinuities in the product's payoff might be optimal from the issuer's viewpoint (see Bernard et al., 2007). We expect that such products will be more and more sold in the future, it is thus important to be able to price them.

Shark contracts (defined above by formula (14)) are indeed very similar to typical equity-linked securities. For instance, in the prospectus supplement of the EKLS,[3] Citigroup Funding Inc. provides first a general description of this equity-linked

[3]Prospectus available online at www.amex.com, dated November 30, 2005.

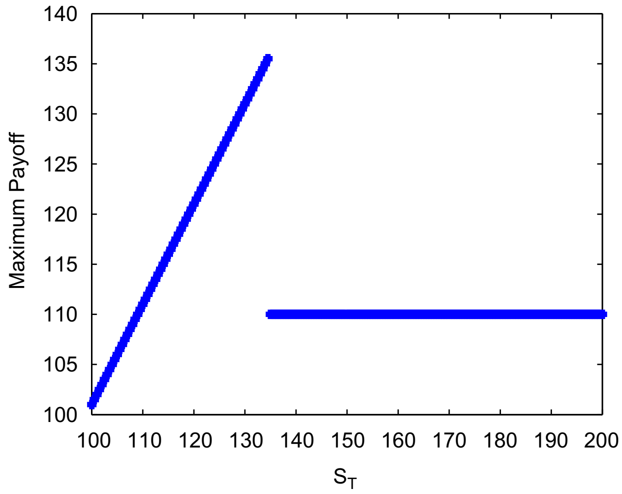

Fig. 1. Payoff of a shark option when the barrier has not been hit before the maturity of the contract. It is thus the maximum cash-flows the investor might expect at expiry time. Parameters are set to $\beta = 1.1$, $M = 100$, $H = (1+\alpha)S_0 = 135$.

security: "at maturity, the EKLS return either the principal amount of your investment in cash or, if the stock on which they are based declines by a predetermined percentage or more at any time after the date of the prospectus supplement (...) up to and including the third trading day before maturity (...) a fixed number of shares of the underlying stock on which they are based." Their maturity payments are driven by a down and in event. The shark product presented above is determined by an up and out event. Obviously the presented methodologies apply in both cases. Let us now come back to our example and precise notation.

Denoting by r the risk-free interest rate, and using the fundamental result of arbitrage pricing theory, and the expression of the final payoff (15), we can express the shark's option arbitrage-free price at time 0 as

$$C(0,T) = E_Q\left(\mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}\left(1 + \frac{1}{S_0}(S_T - S_0)^+ \mathbb{1}_{S_{\max} \leqslant H} + (\beta - 1)\mathbb{1}_{S_{\max} > H}\right)\right). \tag{17}$$

Coming now to the practical valuation of our barrier product, we set ourselves in the forward-neutral world where the underlying follows (6). One readily obtains using this latter world:

$$\frac{C(0,T)}{P(0,T)} = \left(1 + \frac{1}{S_0} C^{\mathrm{uo}}(S_T, K = S_0, \text{Barrier } H) + (\beta - 1) Q_T(S_{\max} > H)\right). \tag{18}$$

The only term that cannot be computed using the first section is $Q_T(S_{\max} > H)$. We denote it by $\mathscr{E}$ and this is in fact $Q_T(\gamma^{\mathrm{u}} \leqslant T)$. Using the approximation of the distribution of γ^{u} (see Appendix A), one obtains

$$\mathscr{E} = Q_T(\gamma^{\mathrm{u}} \leqslant T) \approx \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} q^{\mathrm{u}}(i,j).$$

Indeed, to obtain this formula, one should start writing

$$\mathscr{E} = \int_0^T \int_{-\infty}^{+\infty} Q_T\big(r_{\gamma^{\mathrm{u}}} \in \mathrm{d}r, \gamma^{\mathrm{u}} \in \mathrm{d}s\big)$$

and then discretize along time and interest rate, and introduce the $q^{\mathrm{u}}(i,j)$ terms.

Using this discretized version of the first-passage time distribution, one can obtain the following formula for the shark contract value when the barrier is constant:

$$C(0,T) = P(0,T) + \frac{1}{S_0} C^{\mathrm{uo}}(S_T, K = S_0, H) + (\beta - 1)P(0,T)\mathscr{E}$$

which can be computed straightforwardly using results from Section 1.2. We shall now concentrate on the particular case where the barrier is slightly modified in terms of a zero-coupon bond: this case is particularly interesting because fully closed-form formulas can be obtained.

3.2. Discounted barrier options

In this subsection, we take into account the effect of discounting the barrier. At first look, such a structured product seems difficult to value fully explicitly. In fact, we show below that this is the contrary and that the pricing problem can be solved in closed-form. We assume that the frontier is given by a discounted constant barrier. Formally, K being a constant, the barrier is a stochastic process $(D_t)_{t\in[0,T]}$ such that

$$D_t = KP(t,T), \tag{19}$$

where, only in this section, this expression replaces in the contract covenant the barrier $H = (1+\alpha)S_0$. The shark's formula then becomes

$$C(0,T) = P(0,T)E_{Q_T}[(1+R_T)\mathbb{1}_{\{\forall t\in[0,T], S_t \leqslant D_t\}} + \beta \mathbb{1}_{\{\exists t\in[0,T], S_t > D_t\}}].$$

In fact, the factor $1 + R_T = 1 + (S_T - S_0)^+/S_0$ can also be written as:

$$1 + R_T = \mathbb{1}_{\{S_T < S_0\}} + \frac{S_T}{S_0}\mathbb{1}_{\{S_T > S_0\}} \tag{20}$$

which allows to write together with Eq. (19):

$$\begin{aligned} C(0,T) = \beta P(0,T) Q_T&\left(\sup_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right) > K\right) \\ &+ P(0,T)Q_T\left(S_T < S_0, \sup_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right) \leqslant K\right) \\ &+ E_Q\left[\mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}\frac{S_T}{S_0}\mathbb{1}_{\{S_T \geqslant S_0, \sup_{0\leqslant t\leqslant T}(S_t/P(t,T)) \leqslant K\}}\right]. \end{aligned}$$

Notice that the third term is expressed under the risk-neutral probability Q. To simplify the following developments, we divide the Shark contract into a sum of three expressions according as

$$C[0,T] = P(0,T)[\beta E_1 + E_2] + E_3,$$

where the three sub-contributions to the contract can be defined as:

$$\begin{cases} E_1 = Q_T\left(\sup\limits_{0\leqslant t\leqslant T}\left(\dfrac{S_t}{P(t,T)}\right) > K\right), \\ E_2 = Q_T\left(S_T < S_0,\ \sup\limits_{0\leqslant t\leqslant T}\left(\dfrac{S_t}{P(t,T)}\right) \leqslant K\right), \\ E_3 = E_Q\left[\mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}\dfrac{S_T}{S_0}\mathbb{1}_{\{S_T\geqslant S_0, \sup_{0\leqslant t\leqslant T}(S_t/P(t,T))\leqslant K\}}\right]. \end{cases} \tag{21}$$

Then, against all expectations, one can obtain the following proposition:

Proposition 3.1. *The three components of a shark contract, when the barrier is proportional to a zero-coupon bond and under a Vasicek term structure of interest rates, can be written in closed-form as follows*:

$$\begin{cases} E_1 = \mathcal{N}\left(\dfrac{\ln\left(\dfrac{S_0}{KP(0,T)}\right) - \dfrac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) + \dfrac{S_0}{KP(0,T)}\mathcal{N}\left(\dfrac{\ln\left(\dfrac{S_0}{KP(0,T)}\right) + \dfrac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right), \\ E_2 = \mathcal{N}\left(\dfrac{\ln(P(0,T)) + \dfrac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \dfrac{S_0}{KP(0,T)}\mathcal{N}\left(\dfrac{\ln\left(\dfrac{S_0^2}{K^2P(0,T)}\right) + \dfrac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right), \\ E_3 = \mathcal{N}\left(\dfrac{\ln\left(\dfrac{KP(0,T)}{S_0}\right) - \dfrac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \dfrac{KP(0,T)}{S_0}\mathcal{N}\left(\dfrac{\ln\left(\dfrac{S_0}{KP(0,T)}\right) - \dfrac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) \\ \qquad - \mathcal{N}\left(\dfrac{\ln(P(0,T)) - \dfrac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) + \dfrac{KP(0,T)}{S_0}\mathcal{N}\left(\dfrac{\ln\left(\dfrac{S_0^2}{K^2P(0,T)}\right) - \dfrac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right), \end{cases} \tag{22}$$

where $\tau(T) = \int_0^T [(\sigma_P(u,t) + \rho\sigma)^2 + \sigma^2(1-\rho^2)]\,\mathrm{d}u$ *and* $\mathcal{N}$ *is the cumulative standard normal distribution function.*

The proof of this proposition can be found in Appendix C. To sum up, we have obtained a closed-form formula for the shark option in the case of a stochastic barrier defined as in (19). Moreover, this closed-form formula is very simple and has the same computational efficiency as the one we would obtain with a constant term structure of interest rates (see Rubinstein and Reiner, 1991 for the pricing of barrier options in a Black and Scholes context).

Unfortunately, the simplicity of the above result does not hold when the barrier is merely a constant one, as exposed in the beginning of this article: semi-closed form formulas are then in order. In the coming section, we shall compare the two main contracts (defined, respectively, with a discounted and a constant barrier); a full sensitivity analysis of these products will be presented.

4. Numerical analysis

One of the main goals of this article being to develop a new methodology to study barrier products in the presence of stochastic interest rates, we start by checking its accuracy by comparing the results it provides to the ones obtained by means of Monte-Carlo simulations. By doing so, we show that the extended Fortet method does indeed work correctly, and that it is much faster than the Monte-Carlo method.

Secondly, and from Section 3.3 on, we shall concentrate on the analysis of the shark option, which is the core product example of our study. We compare the prices and sensitivities of these contracts written either with a stochastically discounted barrier, e.g. $(1+\alpha)S_0P(t,T)$, or with a constant barrier, e.g. $(1+\alpha)S_0$. Amongst the sensitivities studied here are the ones computed with respect to the barrier level, to the underlying index's volatility or to its correlation with the interest rates. Let us start by giving the values of the parameters involved in our numerical analysis.

4.1. Parameters

In Table 1, we give some values for the general parameters useful for the coming option valuations. Some of them will vary later on, and this shall be indicated in due time.

We briefly recall the meanings of the above coefficients. The nominal of the contract, M, is set to one for the sake of simplicity. S_0 stands for the initial value of the Equity Index. σ is the underlying's volatility and is set to 20%. The contract's maturity, T, is equal to 1 year. As concerns the maximum yield, in other words the factor governing the level of the barrier, it is given by $1+\alpha=1.35$. The barrier level is indeed given by $H=(1+\alpha)S_0=135$. The rebate's percentage is equal to $\beta=110\%$. ρ is the correlation coefficient between the index process and the instantaneous interest rate process r. We made our study with an exponential

Table 1
Data

M	S_0	σ	T	α	β	ρ
1	100	20%	1	0.35	1.1	0.3

Table 2
Interest rate process

a	v	r_0	θ
0.46	0.007	0.015	0.05

Table 3
Shark option values, $r_{\min} = 0$ and $r_{\max} = 0.3$

Extended Fortet		Shark price	Time (in seconds)
n_T	n_r		
50	50	1.0347	5
100	100	1.0296	95
120	120	1.0289	216
150	150	1.0282	598
180	180	1.0278	1291
200	200	1.0276	3881

structure of the volatility of the interest rates, specified by the two parameters a and v. The values chosen for the interest rate process parameters are given in Table 2.

r_0 and θ are necessary to specify the initial term structure of interest rates. In the particular calibration subsetting chosen here, this is equivalent to knowing the Government yield curve.

4.2. Pricing and hedging with the extended Fortet method

We first look at the pricing method and its accuracy, and then we concentrate on the computation of Greeks.

4.2.1. Pricing issues

We start by pricing a shark option when the main parameters are defined as in Tables 1 and 2. Table 3 displays numerical estimations of the option, done with the extended Fortet method, and discretizing the interest rate between 0 and 0.3 (30% is a very superior bound for an interest rate in a developed country). The extended Fortet method is very fast. We observe that 150 discretization steps for the interest rate and time already give an accurate result, estimating an asymptotic result of about $1 \cdot 0275$.

Then, in Table 4, we compute the price of the same product, under the same conditions, but we discretize the interest rate between -5% and 30%. Of course an interest rate of -5% has no empirical meaning or existence. We display this table for two reasons. First, it is often argued that Gaussian interest rate models permit negative interest rates. In fact, this depends a lot on the calibrated parameters of the driving process. In the current setting, we can observe by comparing Tables 3 and 4

Table 4
Shark option values, $r_{\min} = -0.05$ and $r_{\max} = 0.3$

Extended Fortet		Shark price	Time (in seconds)
n_T	n_r		
50	50	1.0356	5
100	100	1.0299	85
120	120	1.0291	207
150	150	1.0283	544
180	180	1.0279	1168
200	200	1.0276	3009

that allowing for negative interest rates in the grid only modifies the results by one basis point (as soon as we have a reasonable degree of discretization of 150×150), and that asymptotically the two results are the same. Second, and consequently, this shows that the results are quickly not sensitive to the bounds imposed on the interest rate, provided these bounds are reasonably large.

For the sake of brevity, we do not report here other tables, done with other values of $r_{\min}$ and $r_{\max}$. The conclusions are the same. Namely, one can discretize the interest rate between 0% and 20% or less, and obtain accurate results. A discretization grid of 200×200 points in time and interest rate gives a good estimation of the price. This can be confirmed by performing Monte-Carlo simulations, although they take much more time to compute. Indeed, and not surprisingly, this path-dependent problem requires with Monte-Carlo a very thin time discretization and many sample paths, due in particular to the bias that typically appears when valuing a barrier by means of simulations (the question of *ad hoc* corrections is discussed afterwards). To conclude, the extended Fortet method computes more quickly than rough, uncorrected Monte-Carlo simulations based on Euler schemes.

The goal of this paper is not the acceleration of the extended Fortet method, or of Monte-Carlo simulations. Nevertheless, we believe it can be useful to discuss the corrections needed for both methods. We start with the problems traditionally associated with Monte-Carlo simulations, problems that are quite well known.

When performing Monte-Carlo simulations, setting a time step small enough is of critical importance for the precision of the evaluation. If not, a discretization bias shifts the value of the contract, whatever the number of simulations. To enhance the accuracy of the method, Broadie et al. (1997) propose to shift the level of the barrier (their proofs are done in the Black and Scholes framework). It cannot be easily applied to our setting that includes stochastic interest rates. Howison and Steinberg (2007) show how to extend this continuity correction to a wide variety of contracts and models and compute higher order terms in the correction by using the match asymptotic expansion (details and other applications can also be found in Howison, 2005). An other alternative to increasing the number of time steps is to use a method correcting for the bias induced by the hitting probabilities between two time steps:

see for instance the paper of Andersen and Brotherton-Ratcliffe (1996). This method, also called the 'continuity correction', works well in the framework of Black and Scholes (note in passing that this method can also be implemented in a few particular settings with jumps, see for example Ribeiro and Webber, 2003). For each path, one has to correct the fact that the barrier might have been triggered between two steps of time discretization. We cannot apply directly this enhancement to improve the Monte-Carlo simulations in our case since the interest rates are stochastic and we do not have a closed-form formula for the probability to cross the barrier between two steps.

Such corrections cannot apply to improve the extended Fortet method because it is based on the construction of a time grid for the stopping time, and not on the sampling of trajectories as with the Monte-Carlo method. Also, the computation of conditional expectations, in this Fortet setting, is based on the approximation of an integral by the '*rectangle method*', which amounts to approximating the function to integrate by a piecewise constant function and then to integrating the latter. Note that several methods can be used to improve the convergence of the rectangle method. First and instead, it is possible to use the '*trapezoidal rule*' which converges a little bit faster. Second, it is well known that using Gaussian quadratures improves the speed of convergence. Gaussian quadratures provide flexibility in choosing not only the weighting coefficients (weight factors) but also the locations (abscissas) where the functions are evaluated. As a result, Gaussian quadratures yield twice as many places of accuracy as that of the Newton–Cotes formulas with the same number of function evaluations. When the function is known and smooth, Gaussian quadratures usually have a decisive advantage in efficiency. This amounts to writing

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{k=1}^{n} w(x_k)f(x_k) + R_n(x),$$

where x_k are the zeros of orthogonal polynomials. They are the integration points and $w(x)$ is the weighting function related to the orthogonal polynomials; R_n is the error term. We refer the reader to textbooks on numerical integration for more details.

4.2.2. Hedging issues

In general, investors are not only facing pricing issues but are also highly interested in managing their portfolios and immunizing them with respect to the various variables of the market (underlying's volatility, index value, interest rate, etc.). In order to do that, they need to be able to estimate the sensitivities of derivatives' prices (commonly referred to as 'Greeks'). Remark in passing that barrier products are known to be difficult to hedge since the sensitivity to the underlying becomes infinite when the underlying is close to the barrier.

When using Monte-Carlo methods, the first idea, in order to compute Greeks, is to use 'finite-difference approximations', but this produces biased estimates. Alternatives are proposed by Glasserman (2003). First, he proposes the pathwise method which consists in differentiating each simulated payoff with respect to the parameter of interest. But discontinuities in the payoff are a main obstacle to the applicability of this method. It seems to be impossible to get accurate results for barrier options using this technique. In our case, the second method he proposes seems to be more appropriate. It is called the 'likelihood ratio' method and amounts to differentiating a probability density rather than an outcome. But then we need to know the underlying density. In our particular problem, the density involved is the joint distribution of (τ, S_τ) which is unknown as mentioned previously. Thus it might also be difficult to apply this approach. Moreover, sensitivities obtained by Monte-Carlo simulations are known to be not smooth and to converge very slowly. To conclude, computing Greeks with Monte-Carlo simulations is a difficult issue when dealing with path-dependent products.

Two main advantages of the extended Fortet approach are the quickness of convergence, and the fact that we do not need to smooth Greeks (and prices). When the steps $\delta t = T/n_T$ and $\delta r = (r_{\max} - r_{\min})/n_r$ become smaller, estimates become more and more precise. We can compute Greeks in a very simple way. Assume one needs to compute the sensitivity of the price with respect to any factor denoted by x. Consider two close values x_1 and x_2 of x. The price, P, is computed for each value of x. So the Greek w.r.t. x can be estimated by $(P(x_2) - P(x_1))/(x_2 - x_1)$. One simply has to make sure that x_1 and x_2 are close enough. This is the extremely simple method with which many Greeks like the Vega and Rho and computed when using lattice methods. This method, although simple, works particularly well in the extended Fortet method, as the following developments illustrate.

Figs. 2–7 give some examples of prices and Greeks. Fig. 2 represents the price of a shark option, with the set of parameters given in Section 4.1. The derivative with respect to the underlying index S_0, or Delta of the option, is represented in Fig. 3.

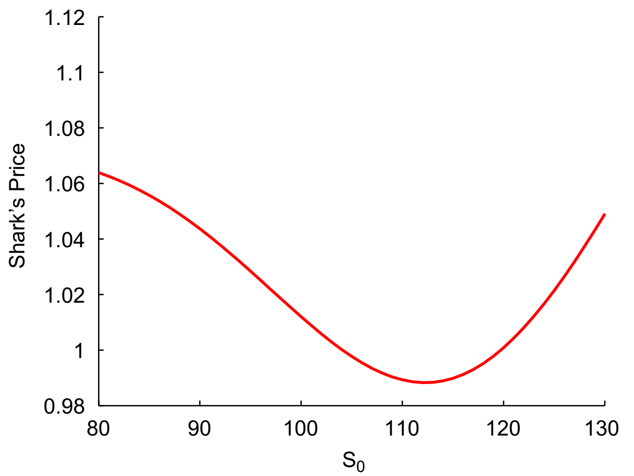

Fig. 2. Shark price w.r.t. S_0.

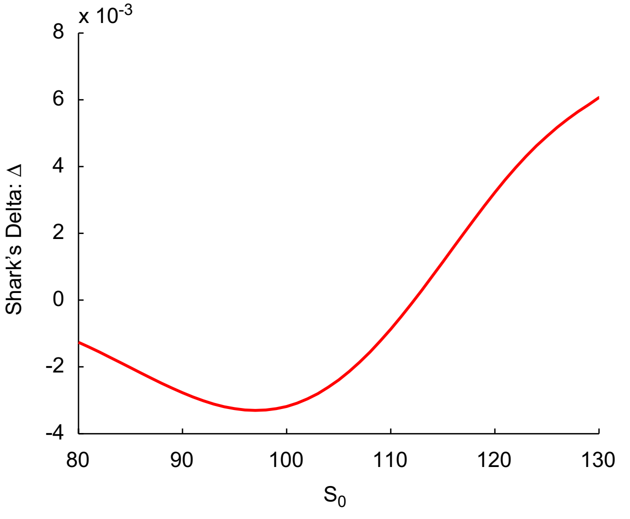

Fig. 3. Delta of the shark w.r.t. S_0.

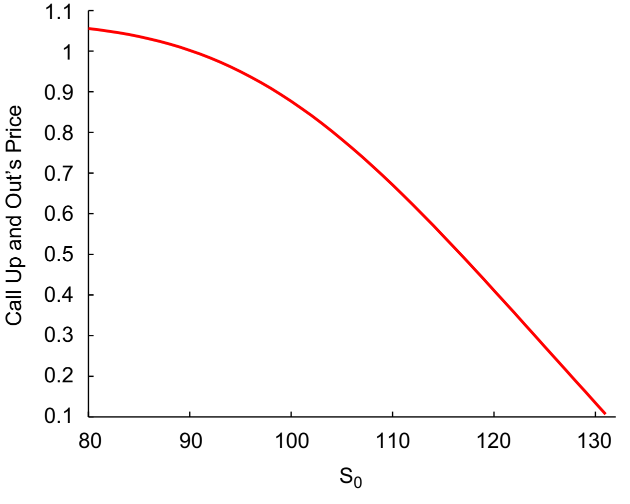

Fig. 4. Up and out call price w.r.t. S_0.

The price of the option is not monotonic with respect to the index because this option has a rebate. If we set $\beta = 0$, the shark option becomes a standard up and out call option (UOC) whose price and Delta are represented in Figs. 4 and 5; these graphs possess familiar shapes. We also plot the Gamma of the up and UOC and of the Shark with respect to S_0 in Figs. 6 and 7. Note that the presence of a rebate sharply changes the behavior of the price, Delta and Gamma of the option with respect to the underlying's initial value.

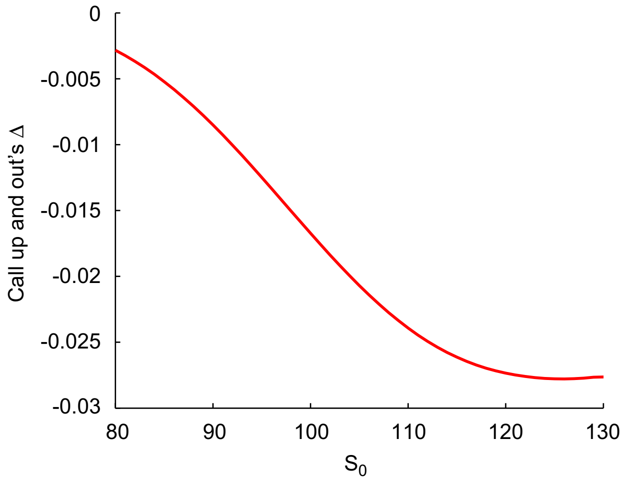

Fig. 5. Delta of the UOC w.r.t. S_0.

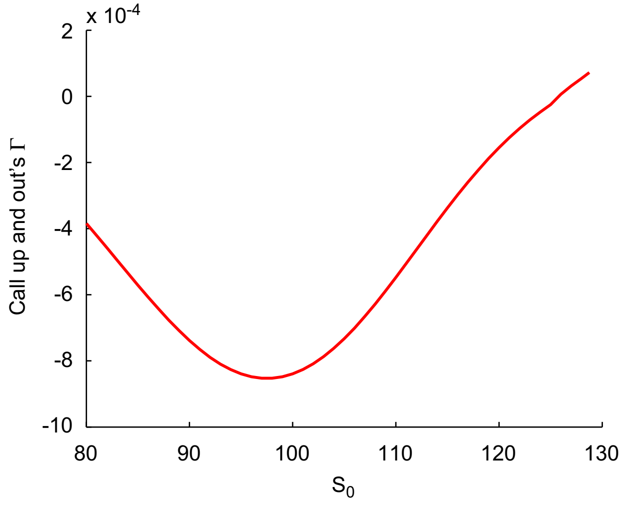

Fig. 6. Gamma of the UOC w.r.t. S_0.

4.3. Comparison of contracts

We want to compare the two types of contracts described in Sections 3.1 and 3.2 (shark contracts with, respectively, a constant barrier H and a discounted barrier $KP(t,T)$). To enable an efficient comparison of both contracts, from now we set $H = (1+\alpha)S_0$ and $K = (1+\alpha)S_0$ (identical levels at contract maturity for both barrier products). We study the sensitivities of both options with respect to the volatility of the underlying index, to the correlation and to the level of the barrier.

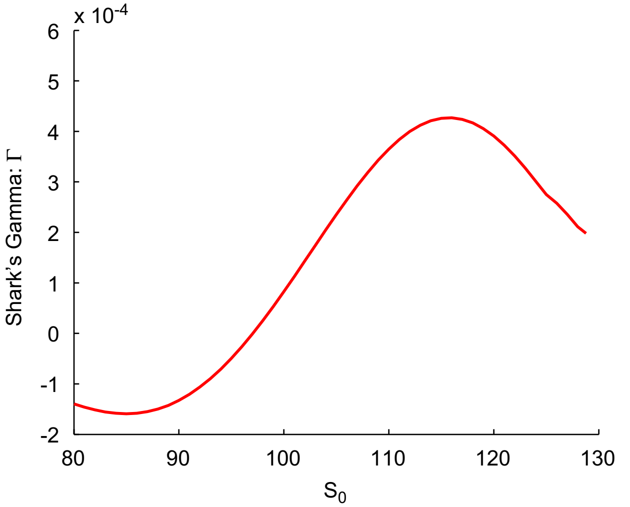

Fig. 7. Gamma of the shark w.r.t. S_0.

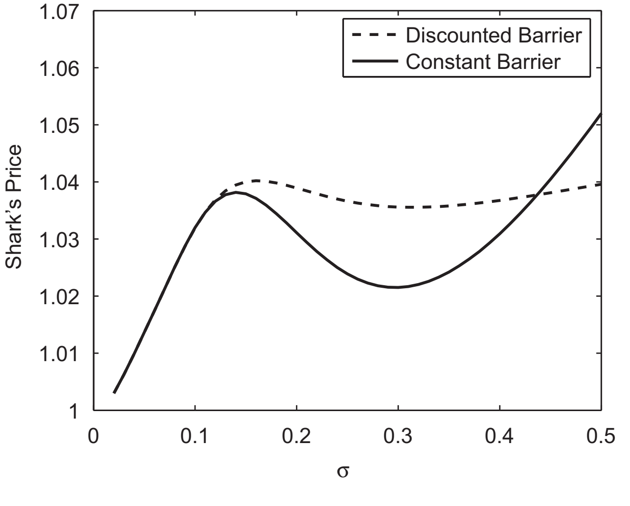

Fig. 8. Shark price w.r.t. σ.

This enlightens interesting properties of the product with the discounted barrier in terms of hedging the sensitivity to interest rates. Note also that the sensitivity to the barrier level is not standard since we do not study standard barrier option but barrier options with a rebate.

4.3.1. *Impact of the index volatility* σ

Let us now come to a brief study of a shark option's dependence on the underlying index volatility. We represent in Fig. 8 the price of a shark with,

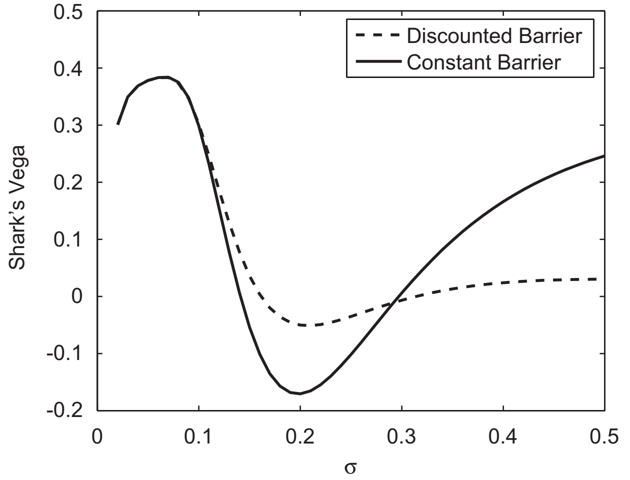

Fig. 9. Vega w.r.t. σ.

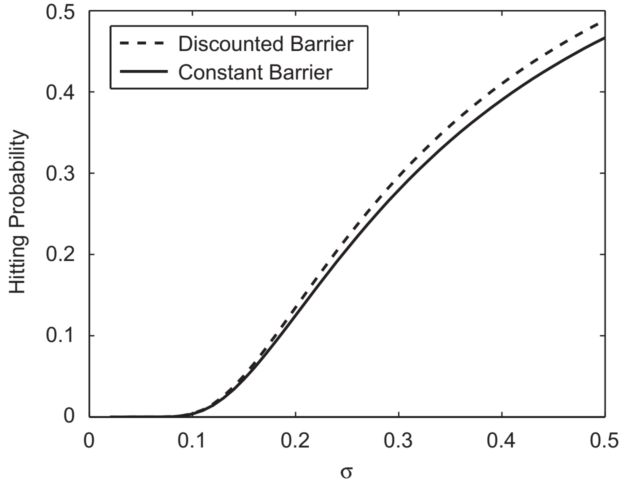

Fig. 10. Hitting probability w.r.t. σ.

respectively, a constant barrier and a discounted barrier, and in Fig. 9 we plot the sensitivities of these prices with respect to the volatility σ of the underlying (also known as Vega). Note that the sensitivities are seen to be smooth, and can also be expected to be reasonably accurate.

For these graphs, all parameters are chosen as in Section 4.1, except σ which ranges between 1% and 80%. One can first observe that the Vega is obtained as a smooth function of the underlying and again that the presence of a rebate changes the standard sensitivity to volatility of barrier products.

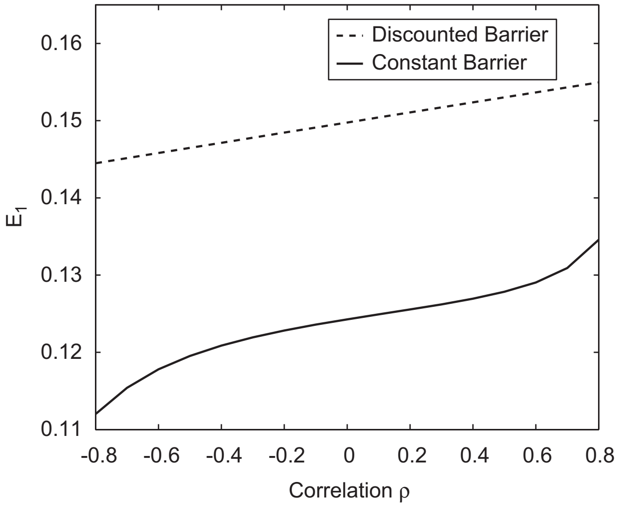

Fig. 11. Hitting probability w.r.t. ρ.

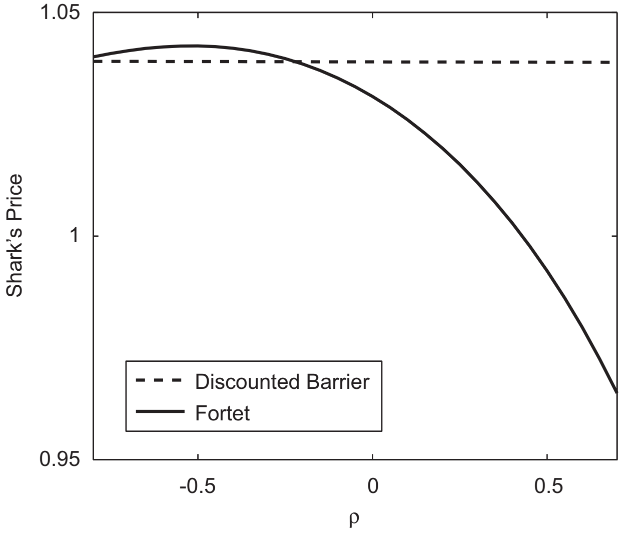

Fig. 12. Shark's price $C(0, T)$ w.r.t. ρ.

4.3.2. *Impact of the correlation ρ*

We plot in Figs. 10 and 11 the probability to hit the barrier (before the contract maturity) with respect to σ, the volatility of the underlying, and with respect to ρ, the correlation coefficient between the Equity Index and the interest rate. For both cases, when the volatility of the underlying increases, the hitting time probability increases accordingly. They have the same behavior with respect to the correlation too. We can notice that the hitting probability is always higher in the case of a discounted barrier and it sharply depends on σ.

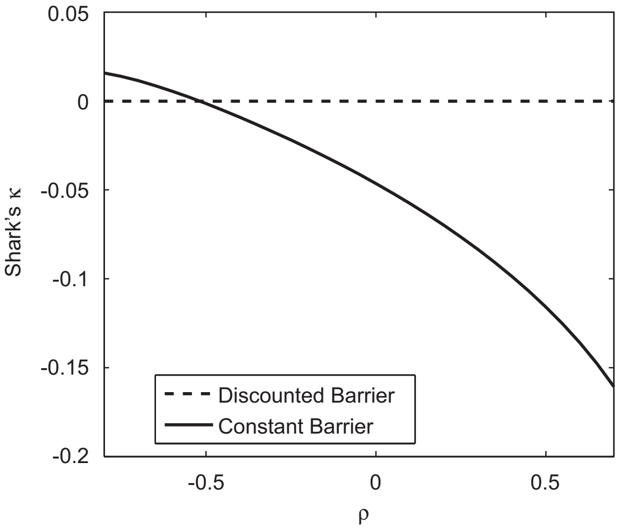

Fig. 13. Shark's kappa w.r.t. ρ.

Figs. 12 and 13 plot, respectively, the contract value with respect to the correlation, and its sensitivity to the correlation (often called 'Kappa' of the option), again w.r.t. the correlation. We let the correlation ρ vary between -0.8 and 0.8 in both graphs (we do this for the sake of illustration, this is not a restriction).

Let us first consider the case of a discounted barrier. In this particular situation, the contract price is nearly insensitive to a change in the correlation. On the contrary, when the barrier is constant, the shark's price is a remarkably decreasing function of the correlation. One of the advantages of imposing a stochastic barrier clearly appears here: it can help cancel the impact of the randomness of interest rates on derivative prices and hedge the correlation between the underlying and interest rates. The behaviors of the two subproducts clearly varies with respect to the correlation. Note that they have similar fluctuations with respect to the volatility.

4.3.3. *Impact of the barrier level*

Let us now come to the numerical study of the dependence on the barrier level. To do this, we plot the probability of hitting the barrier, in Fig. 14, and $C(0,T)$, the contract's value, in Fig. 15 with respect to the barrier level (defined, respectively, by $H=(1+\alpha)S_0$ and $HP(t,T)$). We keep the parameter values from Table 1, except α which ranges between 0.1 and 1 (in correspondence to H which ranges between 110 and 200).

The interpretation of Fig. 14 is straightforward: as the barrier increases, the probability that it be hit sharply diminishes. When the barrier value is high enough (say 170), the probability to reach it is nearly null. Despite the gross appearance of Fig. 15, the influence of the barrier level on the price is indeed quite small. In particular, the contract's price shows a relative variation of less than 3% when the barrier goes from 110 to 200 (assuming $\beta=1.1<1+\alpha$). The explanation of this phenomenon obtains directly from a P&L analysis. At expiry time T, the investor gets back his initial

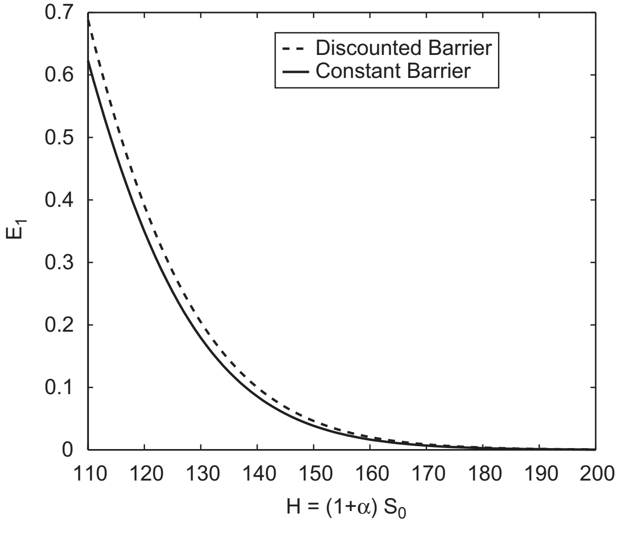

Fig. 14. Hitting probability w.r.t. H.

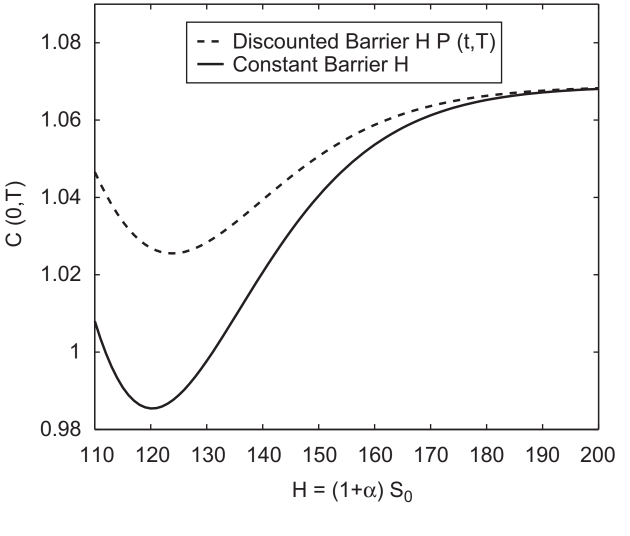

Fig. 15. $C(0,T)$ w.r.t. H.

investment, whether the barrier has been reached or not, and this payment mostly determines the price of the contract. Obviously, for a knock-out option without rebate, we would observe a stronger influence of the barrier on the price.

Let us now come to a finer description of Fig. 15. One can observe that the price of the option is decreasing with respect to the barrier for low levels of the barrier. This comes from the fact that for $\beta = 1.1$, the rebate is quite important. Choosing a rebate $\beta = 0.3$ and *ceteris paribus*, one would obtain the graph displayed in Fig. 16.

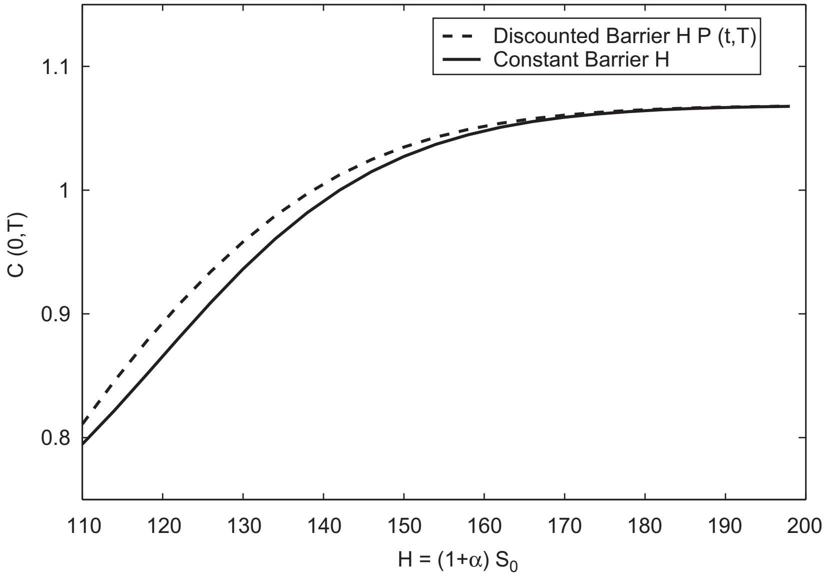

Fig. 16. $C(0,T)$ w.r.t. H with $\beta = 30\%$.

Now, how can we explain the weird behavior of the shark price when the barrier varies between 110 and 120 in Fig. 15? In general, it is advantageous not to hit the barrier; yet, in the presence of a high rebate, say when $1+\alpha \approx \beta$, the probability to get a yield strictly superior to β is equal to the joint probability that the following events occur: $S_{\max} < (1+\alpha)S_0$ and $S_T > \beta S_0$. As this joint probability is very weak, it is in the interest of the optionholder that the barrier be hit, in order to ensure a return at least equal to β. To conclude on this particular situation, when the barrier level is increased, the probability to reach it is diminished, and the contract becomes less interesting, which explains the decrease of its price.

5. Conclusion

This article develops a general methodology useful for pricing barrier options in a Vasicek framework. When the derivative's barrier is a discounted one, we show that it is possible to obtain closed-form formulas to price it, using time change techniques. When the barrier is constant, quasi-closed-form formulas can be found. These latter formulas are computed using the extended Fortet method, exposed within a new and clean apparel in the first appendix of this text, and whose first implementation dates back to Collin-Dufresne and Goldstein (2001) in their seminal structural model of the firm. We indeed obtain general formulas that extend the ones of Rubinstein and Reiner (1991) for pricing barrier options when the driving risk-free interest rate is a Vasicek process. We illustrate our approach on a particular exotic derivatives, the shark index, which is indeed a type of up and out barrier option with rebate.

Structured products have become more and more popular on equity and hybrid markets. Long-term barrier products represent a higher and higher percentage of index linked products. The maturities of these products are often around 5 years: being able to compute their sensitivity to interest rate risk is thus of an utmost importance for risk-management purposes. Concluding this paper, a numerical analysis is conducted based on an example of these structured products (namely a shark option). This analysis gives a practical illustration of the extended Fortet method, and we have demonstrated its practical efficiency. We also show how both the pricing and hedging of these products can be done, and in fact our results hold for all standard barrier options (their full pricing formulas being given in the first section of this paper).

Acknowledgments

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Appendix A. The extended Fortet method

Let us assume that one initially observes $\ln(A_0) = l_0 > \ln(H) = h$. The process l_t is continuous. If at time t, the process $l_t = \ell < h$ then the barrier has been hit and the *down* condition is realized. We denote by γ^{d} this first hitting time. Thanks to this remark, one has

$$
\begin{aligned}
&Q_T(l_t \in [\ell, \ell + \mathrm{d}\ell[, r_t \in [r, r + \mathrm{d}r[\,|\, l_0, r_0) \\
&\quad = \int_0^t \int_{-\infty}^{+\infty} Q_T(l_t \in [\ell, \ell + \mathrm{d}\ell[, r_t \in [r, r + \mathrm{d}r[\,|\, l_s = h, r_s = r') \\
&\quad \times Q_T\begin{pmatrix} r_{\gamma^{\mathrm{d}}} \in [r', r' + \mathrm{d}r'[\\ \gamma^{\mathrm{d}} \in [s, s + \mathrm{d}s[\end{pmatrix}.
\end{aligned}
$$

Let us integrate the previous equation with respect to ℓ between $-\infty$ and h. Using Fubini's theorem, one obtains

$$
\begin{aligned}
&Q_T(l_t \leqslant h, r_t \in [r, r + \mathrm{d}r[\,|\, l_0, r_0) \\
&\quad = \int_0^t \int_{-\infty}^{+\infty} Q_T(l_t \leqslant h, r_t \in [r, r + \mathrm{d}r[\,|\, l_s = h, r_s = r') \\
&\quad \times Q_T\begin{pmatrix} r_{\gamma^{\mathrm{d}}} \in [r', r' + \mathrm{d}r'[\\ \gamma^{\mathrm{d}} \in [s, s + \mathrm{d}s[\end{pmatrix}.
\end{aligned}
\tag{A.1}
$$

To simplify the notations, we define, respectively, Φ and Ψ by

$$\begin{cases} \Phi(r,t)\,\mathrm{d}r = Q_T(l_t \leqslant h, r_t \in [r, r+\mathrm{d}r[\,|\, l_0, r_0), \\ \Psi(r,t,r',s)\,\mathrm{d}r = Q_T(l_t \leqslant h, r_t \in [r, r+\mathrm{d}r[\,|\, l_s = h, r_s = r'). \end{cases}$$

Under these assumptions, Φ and Ψ could be expressed as closed-form formulas. The previous Eq. (A.1) becomes

$$\Phi(r,t) = \int_{s\in[0,t]} \int_{r'\in\mathbb{R}} \Psi(r,t,r',s) Q_T(r_{\gamma^{\mathrm{d}}} \in [r', r'+\mathrm{d}r'[, \gamma^{\mathrm{d}} \in [s, s+\mathrm{d}s[). \tag{A.2}$$

As the distribution function of γ^{d} is unknown, we approximate it. Discretizing along time and interest rate, with n_T discretization steps for the time $(t_0 = 0, t_1, \ldots, t_{n_T} = T)$ and n_r for the interest rate. One has $r_0 = r_{\min}, \ldots, r_{n_r} = r_{\max}$ where $r_{\min}$ and $r_{\max}$ are chosen such as the probability that r takes values outside the interval $[r_{\min}, r_{\max}]$ is negligible. We denote by $q^{\mathrm{d}}(i,j)$:

$$q^{\mathrm{d}}(i,j) = Q_T(r_{\gamma^{\mathrm{d}}} \in [r_i, r_{i+1}], \gamma^{\mathrm{d}} \in [t_j, t_{j+1}]),$$

where the superscript d is for a *down* barrier.

Then, formula (A.2) could be written as

$$\Phi(r_i, t_j) = \sum_{v=0}^{j} \sum_{u=0}^{n_r} \Psi(r_i, t_j, r_u, t_v) q^{\mathrm{d}}(u, v).$$

If $j = 0$, the previous expression becomes

$$\Phi(r_i, t_0) = \sum_{u=0}^{n_r} \Psi(r_i, t_0, r_u, t_0) q^{\mathrm{d}}(u, 0).$$

We then obtain the following expression: $q^{\mathrm{d}}(i,0) = Q_T(r_{\gamma^{\mathrm{d}}} \in [r_i, r_{i+1}], \gamma^{\mathrm{d}} \in [t_0, t_1])$. Noting that $\Psi(r_i, t_0, r_u, t_0) = \mathbb{1}_{\{r_i = r_u\}}$, one readily has

$$q^{\mathrm{d}}(i,0) = \Phi(r_i, t_0).$$

The quantities $q^{\mathrm{d}}(i,j)$ can be computed by means of a recursive algorithm. First, the quantities $q^{\mathrm{d}}(i,0)$ are computed for every i thanks to the above expression. From them the quantities $q^{\mathrm{d}}(i,j)$ for $j \geqslant 1$ are recursively obtained.

$$\begin{aligned} \Phi(r_i, t_j) &= \sum_{v=0}^{j} \sum_{u=0}^{n_r} q^{\mathrm{d}}(u,v) \Psi(r_i, t_j, r_u, t_v) \\ &= \sum_{u=0}^{n_r} q^{\mathrm{d}}(u,j) \Psi(r_i, t_j, r_u, t_j) + \sum_{v=0}^{j-1} \sum_{u=0}^{n_r} q^{\mathrm{d}}(u,v) \Psi(r_i, t_j, r_u, t_v). \end{aligned}$$

Thanks to $\Psi(r_i, t_j, r_u, t_j) = \mathbb{1}_{\{r_i = r_u\}}$, one has

$$q^{\mathrm{d}}(i,j) = \Phi(r_i, t_j) - \sum_{v=0}^{j-1} \sum_{u=0}^{n_r} q^{\mathrm{d}}(u,v) \Psi(r_i, t_j, r_u, t_v). \tag{A.3}$$

To sum up, we have now, with formula (A.3) the possibility to compute the terms $q^{\mathrm{d}}(i,j)$, which give us the approximated distribution function of γ^{d} we are looking for because we have closed-form expressions for $\Phi(r,t)$ and $\Psi(r,t,r',s)$:

$$\begin{cases} \Phi(r,t)\,\mathrm{d}r = Q_T(l_t \leqslant h, r_t \in \mathrm{d}r \mid l_0, r_0), \\ \Psi(r,t,r',s)\,\mathrm{d}r = Q_T(l_t \leqslant h, r_t \in \mathrm{d}r \mid l_s = h, r_s = r'). \end{cases}$$

Note that $X=(l,r)$ is a Gaussian Markov process whose dynamics are given by

$$\mathrm{d}X_t = \mathrm{d}\begin{bmatrix} l_t \\ r_t \end{bmatrix} = \begin{bmatrix} r_t - r_g - \dfrac{\sigma^2}{2} - \sigma\rho\sigma_P(t,T) \\ a\left(\theta - \dfrac{v}{a}\sigma_P(t,T) - r_t\right) \end{bmatrix}\mathrm{d}t + \begin{bmatrix} \sigma\rho & \sigma\sqrt{1-\rho^2} \\ v & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{d}Z_1^{\mathrm{T}} \\ \mathrm{d}Z_2^{\mathrm{T}} \end{bmatrix}.$$

Denote by f_{l_t,r_t} the density function of (l_t,r_t) under Q_T. Thanks to conditional results, one obtains

$$f_{l_t,r_t}(\ell,r) = f_{r_t}(r) f_{l_t|r_t}(\ell).$$

$\mathscr{F}_0$ and $\mathscr{F}_s$ represent the available information at time 0 and s. Using the Markov property of (l_t,r_t), conditioning by $\mathscr{F}_s$ is like conditioning by (l_s,r_s). One then obtains Ψ and Φ:

$$\begin{cases} \Phi(r,t) = f_{r_t}(r|\mathscr{F}_0)\displaystyle\int_{-\infty}^{h} f_{l_t|r_t}(\ell \mid \mathscr{F}_0)\,\mathrm{d}\ell, \\ \Psi(r,t,r',s) = f_{r_t}(r|\mathscr{F}_s)\displaystyle\int_{-\infty}^{h} f_{l_t|r_t}(\ell \mid \mathscr{F}_s)\,\mathrm{d}\ell. \end{cases}$$

As the process (l_t,r_t) is Gaussian, the conditional law of $l_t|r_t$ knowing the available information at time s is Gaussian. We denote the conditional moments by $\mu(r_t,l_s,r_s)$ and $\Sigma^2(r_t,l_s,r_s)$:

$$\begin{cases} \mu(r_t,l_s,r_s) = E_{Q_T}[l_t \mid \mathscr{F}_s] + \dfrac{Cov(l_t,r_t \mid \mathscr{F}_s)}{Var[r_t \mid \mathscr{F}_s]}(r_t - E_{Q_T}[r_t \mid \mathscr{F}_s]), \\ \Sigma^2(r_t,l_s,r_s) = Var[l_t \mid \mathscr{F}_s] - \dfrac{Cov(l_t,r_t \mid \mathscr{F}_s)^2}{Var[r_t \mid \mathscr{F}_s]}. \end{cases}$$

The above moments are computed in Appendix B. Let $\mathscr{N}$ be the normal standard distribution function. We then obtain

$$\begin{cases} \Phi(r,t) = f_{r_t}(r|r_0)\mathscr{N}\left(\dfrac{h-\mu(r,l_0,r_0)}{\sqrt{\Sigma^2(r,l_0,r_0)}}\right), \\ \Psi(r,t,r',s) = f_{r_t}(r|r_s=r')\mathscr{N}\left(\dfrac{h-\mu(r,l_s=h,r')}{\sqrt{\Sigma^2(r,l_s=h,r')}}\right), \end{cases}$$

where f_r is the transition density of r. Recall that

$$f_{r_t}(r \mid r_s) = \frac{1}{\sqrt{2\pi v}} e^{-(r-m)^2/2v},$$

where $m = E[r_t|r_s]$ and $v = Var[r_t|r_s]$ (given in Appendix B).

Remark. The *up* case.

The *up* case is in fact the case when $l_0 < h$. We define as γ^{u} the first hitting time of the process l_t to the barrier's level $\ln(H) = h$. The proof is exactly the same as in the *down* case. Thus, one obtains the following formulas for the approximate density of $(r_{\gamma^{\mathrm{u}}}, \gamma^{\mathrm{u}})$ (similar to formula (A.3)):

$$\begin{cases} q^{\mathrm{u}}(i,0) = \Phi^{\mathrm{u}}(r_i, t_0), \\ q^{\mathrm{u}}(i,j) = \Phi^{\mathrm{u}}(r_i, t_j) - \displaystyle\sum_{k=0}^{j-1}\sum_{l=0}^{n_r} q^{\mathrm{u}}(l,k)\Psi^{\mathrm{u}}(r_i, t_j, r_l, t_k), \end{cases} \tag{A.4}$$

where

$$\begin{cases} \Phi^{\mathrm{u}}(r,t) = f_{r_t}(r \mid r_0)\mathcal{N}\left(\dfrac{\mu(r, l_0, r_0) - h}{\sqrt{\Sigma^2(r, l_0, r_0)}}\right), \\ \Psi^{\mathrm{u}}(r,t,r',s) = f_{r_t}(r \mid r_s = r')\mathcal{N}\left(\dfrac{\mu(r, l_s = h, r') - h}{\sqrt{\Sigma^2(r, l_s = h, r')}}\right). \end{cases}$$

Appendix B. Moments of the processes r_t and l_t

We work under the forward-neutral measure Q_T. We compute in this appendix the moments of the instantaneous interest rate r and those of l associated with the index process. We choose to do the study with the exponential structure of volatility. With $v > 0$ and $a > 0$, the volatility structure can be written as follows:

$$\sigma_P(t,T) = \frac{v}{a}(1 - e^{-a(T-t)}).$$

Define B_a by

$$B_a(u) = \frac{1}{a}(1 - e^{-au}).$$

Under the forward-neutral measure, the interest rate process r follows the dynamics given by

$$dr_t = a(\theta_t - r_t)\,dt + v\,dZ_1^{\mathrm{T}}(t),$$

where $\theta_t = \theta - (v^2/a)B_a(T-t)$. Thanks to Itō's lemma and an integration by parts, one obtains

$$r_t = \mathrm{e}^{-at}\left(r_u \mathrm{e}^{au} + \int_u^t \theta_s \mathrm{e}^{as}\,\mathrm{d}s + v\int_u^t \mathrm{e}^{as}\,\mathrm{d}Z_1^{\mathrm{T}}(s)\right).$$

In this particular case, the instantaneous interest rate r is an Ornstein–Uhlenbeck process under the forward-neutral probability Q_T. The zero-coupon bond maturing at T satisfies the relationship:

$$P(t,T) = \mathrm{e}^{-B_a(T-t)r_t - \eta(T-t)}, \tag{B.1}$$

where

$$\eta(u) = \left(\theta - \frac{v^2}{2a^2}\right)(u - B_a(u)) + \frac{v^2}{4a}(B_a(u))^2.$$

Conditional moments of the process r: r is a Gaussian process with the following conditional moments (with $s<t$):

$$\begin{cases} E_{Q_T}[r_t \mid r_u] = \mathrm{e}^{-a(t-u)}r_u + \left(\theta a - \dfrac{v^2}{a}\right)B_a(t-u) + \dfrac{v^2}{a}\mathrm{e}^{-a(T-t)}B_{2a}(t-u), \\ Var_{Q_T}[r_t \mid r_u] = v^2 B_{2a}(t-u), \\ Cov_{Q_T}(r_s, r_t|r_u) = \dfrac{v^2}{2a}\mathrm{e}^{-a(s+t)}(\mathrm{e}^{2as} - \mathrm{e}^{2au}) = v^2\mathrm{e}^{-a(t-s)}B_{2a}(s-u). \end{cases}$$

Conditional moments of the process l: Integrating the dynamics (7) of the process l under Q_T between u and t, one has

$$\begin{aligned} l_t = l_u &+ \int_u^t r_s\,\mathrm{d}s - \left(\frac{\sigma^2}{2} + \frac{\sigma\rho v}{a}\right)(t-u) + \sigma\rho v\int_u^t \mathrm{e}^{-a(T-s)}\,\mathrm{d}s \\ &+ \sigma\rho\int_u^t \mathrm{d}Z_1^{\mathrm{T}}(s) + \sigma\sqrt{1-\rho^2}\int_u^t \mathrm{d}Z_2^{\mathrm{T}}(s). \end{aligned}$$

Now remark that the integral $\int_u^t r_s\,\mathrm{d}s$ is also a Gaussian process whose conditional moments are given by the following formulas:

$$\begin{cases} E_{Q_T}\left[\displaystyle\int_u^t r_s\,\mathrm{d}s \mid \mathscr{F}_u\right] = r_u B_a(t-u) + \displaystyle\int_u^t \mathrm{e}^{-as}\int_u^s \mathrm{e}^{ax}\theta_x\,\mathrm{d}x\,\mathrm{d}s, \\ Var_{Q_T}\left[\displaystyle\int_u^t r_s\,\mathrm{d}s \mid \mathscr{F}_u\right] = \dfrac{v^2}{a^2}(t-u+B_{2a}(t-u) - 2B_a(t-u)), \\ Cov_{Q_T}\left(\displaystyle\int_u^t r_v\,\mathrm{d}v, \int_u^t \mathrm{d}Z_1^{\mathrm{T}}(s) \mid \mathscr{F}_u\right) = \dfrac{v}{a}(t-u-B_a(t-u)). \end{cases}$$

This enables us to obtain the following conditional moments for the process l_t when $s<t$:

$$\begin{cases} E_{Q_T}[l_t\,|\,\mathscr{F}_u] = l_u - \left(r_g + \dfrac{\sigma^2}{2} + \dfrac{\sigma\rho v}{a} - \theta + \dfrac{v^2}{a^2}\right)(t-u) - \dfrac{v^2}{a^2}\mathrm{e}^{-a(T-t)}B_{2a}(t-u) \\ \qquad + \left(r_u - \theta + \dfrac{v^2}{a^2} + \dfrac{v^2}{a^2}\mathrm{e}^{-a(T-t)} + \dfrac{\sigma\rho v}{a}\mathrm{e}^{-a(T-t)}\right)B_a(t-u), \\ Var_{Q_T}[l_t\,|\,\mathscr{F}_u] = \left(\sigma^2 + \dfrac{v^2}{a^2} + 2\dfrac{\sigma\rho v}{a}\right)(t-u) - 2\left(\dfrac{v^2}{a^2} + \dfrac{\sigma\rho v}{a}\right)B_a(t-u) \\ \qquad + \dfrac{v^2}{a^2}B_{2a}(t-u), \\ Cov(l_s, l_t\,|\,\mathscr{F}_u) = \dfrac{v^2}{a^2}\mathrm{e}^{-a(t-s)}B_{2a}(s-u) + \left(\sigma^2 + \dfrac{2\sigma\rho v}{a} + \dfrac{v^2}{a^2}\right)(s-u) \\ \qquad - \left(\dfrac{v^2}{a^2} + \dfrac{\sigma\rho v}{a}\right)(\mathrm{e}^{-a(t-s)} + 1)B_a(s-u). \end{cases}$$

Covariance between l_t and r_t:

$$Cov_{Q_T}(l_t, r_t\,|\,\mathscr{F}_u) = \left(\frac{v^2}{a} + \rho\sigma v\right)B_a(t-u) - \frac{v^2}{a}B_{2a}(t-u).$$

Moments of the first and second order for the process $l_t = \ln(S_t)$: Replacing u by 0 in the above expressions of the conditional moments of l_t, we obtain the following formulas:

$$\begin{cases} M_{\exp}(t) = \ln\left(\dfrac{S_0}{P(0,t)}\right) + \dfrac{v^2}{4a^3} - \left(\dfrac{v^2}{2a^2} + \dfrac{\rho\sigma v}{a} + \dfrac{\sigma^2}{2}\right)t - \dfrac{v^2}{4a^3}\mathrm{e}^{-2at} \\ \qquad + \left(\dfrac{v^2}{2a^3} + \dfrac{\rho\sigma v}{a^2}\right)\mathrm{e}^{-a(T-t)} - \left(\dfrac{v^2}{a^3} + \dfrac{\rho\sigma v}{a^2}\right)\mathrm{e}^{-aT} + \dfrac{v^2}{2a^3}\mathrm{e}^{-a(T+t)}, \\ V_{\exp}(t) = \left(\sigma^2 + \dfrac{v^2}{a^2} + \dfrac{2\rho\sigma v}{a}\right)t - \dfrac{3v^2}{2a^3} - \dfrac{2\rho\sigma v}{a^2} + \dfrac{2v(v + a\rho\sigma)}{a^3}\mathrm{e}^{-at} - \dfrac{v^2}{2a^3}\mathrm{e}^{-2at}, \\ C_{\exp}(v,t) = -\left(\dfrac{\rho\sigma v}{a^2} + \dfrac{v^2}{a^3}\right) + \left(\sigma^2 + \dfrac{2\rho\sigma v}{a} + \dfrac{v^2}{a^2}\right)v - \dfrac{v^2}{2a^3}\mathrm{e}^{-a(t+v)} \\ \qquad + \left(\dfrac{\rho\sigma v}{a^2} + \dfrac{v^2}{a^3}\right)(\mathrm{e}^{-av} + \mathrm{e}^{-at}) - \left(\dfrac{\rho\sigma v}{a^2} + \dfrac{v^2}{2a^3}\right)\mathrm{e}^{-a(t-v)}. \end{cases}$$

Appendix C. Proof of Proposition 2.1

We show here how one can compute the three terms (depending on the supremum of the underlying process). Our main tool is the Dubins–Schwarz theorem which says that a continuous local martingale (say M) can be represented as a Brownian motion time-changed by the quadratic variation of the continuous local martingale (say $B_{\langle M\rangle}$). Let us denote by N the stochastic integral in formula (5):

$$N_t = \int_0^t (\sigma_P(u,T) + \rho\sigma)\,\mathrm{d}Z_1^{\mathrm{T}}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\,\mathrm{d}Z_2^{\mathrm{T}}(u).$$

Let also τ be its quadratic variation: $\tau(t) = \langle N\rangle_t$. N is a martingale, with $N(0) = 0$, and its quadratic variation satisfies

$$\tau(t) = \int_0^t [(\sigma_P(u,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)]\,\mathrm{d}u.$$

Consequently, we may write (5) as $S_t/P(t,T) = (S_0/P(0,T))\exp[N_t - \tau(t)/2]$.

Computation of E_1: Finally, the expression of E_1, the first term of (21), can be expressed as

$$E_1 = Q_T\left(\sup_{t\in[0,T]}\left\{-\frac{\tau(t)}{2} + N_t\right\} > \ln\left(\frac{KP(0,T)}{S_0}\right)\right).$$

Using the Dubins–Schwarz theorem, N is a τ time-changed Q_T-Brownian motion B. This readily yields

$$E_1 = Q_T\left(\sup_{\tau\in[\tau(0),\tau(T)]}\left\{-\frac{\tau}{2} + B_\tau\right\} > \ln\left(\frac{KP(0,T)}{S_0}\right)\right).$$

Then, armed with the law of the supremum of an arithmetic Brownian motion (see for instance the third chapter of Jeanblanc et al., 2007), we can obtain the closed-form formula:

$$E_1 = \mathcal{N}\left(\frac{-\ln\left(\frac{KP(0,T)}{S_0}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) + \frac{S_0}{KP(0,T)}\mathcal{N}\left(\frac{-\ln\left(\frac{KP(0,T)}{S_0}\right) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right).$$

Computation of E_2: To compute E_2, we start noting that

$$E_2 = Q_T\left(-\frac{\tau(T)}{2} + B_{\tau(T)} < \ln(P(0,T)),\ \sup_{\tau\in[\tau(0),\tau(T)]}\left\{-\frac{\tau}{2} + B_\tau\right\} \leqslant \ln\left(\frac{KP(0,T)}{S_0}\right)\right).$$

Here, the problem is solved using the joint law of an arithmetic Brownian motion and its supremum (see the same reference as above). This yields directly

$$E_2 = \mathcal{N}\left(\frac{\ln(P(0,T)) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \frac{S_0}{KP(0,T)}\mathcal{N}\left(\frac{\ln\left(\frac{S_0^2}{K^2P(0,T)}\right) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right).$$

Computation of E_3: To compute E_3 we recall from Eq. (3) that

$$\exp\left(-\int_0^T r_u \,\mathrm{d}u\right)\frac{S_T}{S_0} = \exp\left(-\frac{\sigma^2 T}{2} + \int_0^T \sigma\rho \,\mathrm{d}\widehat{Z}_1(u) + \int_0^T \sigma\sqrt{1-\rho^2}\,\mathrm{d}\widehat{Z}_2(u)\right).$$

Using Girsanov's Theorem, we know that $\widetilde{Z}_1(u) = \widehat{Z}_1(u) - \sigma\rho u$ and $\widetilde{Z}_2(u) = \widehat{Z}_2(u) - \sigma\sqrt{1-\rho^2}u$ are two standard Brownian motions under the appropriate measure $\widetilde{Q}$ built with the Radon–Nikodym density process:

$$\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q} = \exp\left(-\frac{\sigma^2 T}{2} + \int_0^T \sigma\rho \,\mathrm{d}\widehat{Z}_1(u) + \int_0^T \sigma\sqrt{1-\rho^2}\,\mathrm{d}\widehat{Z}_2(u)\right).$$

After changing the measure, one obtains

$$E_3 = \widetilde{Q}\left(S_T \geqslant S_0, \sup_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right) \leqslant K\right).$$

We need the expressions of $S_t/P(t,T)$ under $\widetilde{Q}$. After changing probability measure in the dynamics (3) and (4) of S_t and $P(t,T)$, we can write

$$\frac{S_t}{P(t,T)} = \frac{S_0}{P(0,T)}\exp\left(\frac{\widetilde{\tau}_t}{2} + H_t\right),$$

where $H_t = \int_0^t (\sigma_P(u,T) + \rho\sigma)\,\mathrm{d}\widetilde{Z}_1(u)) + (\int_0^t \sigma\sqrt{1-\rho^2}\,\mathrm{d}\widetilde{Z}_2(u)$ and $\widetilde{\tau}_t = \langle H\rangle_t$.

Then, one obtains

$$E_3 = \widetilde{Q}\left(\frac{\widetilde{\tau}}{2} + \widetilde{B}_{\widetilde{\tau}} \geqslant \ln(P(0,T)), \sup_{\widetilde{\tau}\in[\widetilde{\tau}(0),\widetilde{\tau}(T)]}\left(\frac{\widetilde{\tau}}{2} + \widetilde{B}_{\widetilde{\tau}}\right) \leqslant \ln\left(\frac{KP(0,T)}{S_0}\right)\right),$$

where $\widetilde{B}$ is a standard $\widetilde{Q}$-Brownian motion. Using the same classical results as for E_2 and noting that $\widetilde{\tau}_t = \langle N\rangle_t = \tau_t$, one finally obtains

$$E_3 = \mathcal{N}\left(\frac{\ln\left(\frac{KP(0,T)}{S_0}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \frac{KP(0,T)}{S_0}\mathcal{N}\left(\frac{\ln\left(\frac{S_0}{KP(0,T)}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \mathcal{N}\left(\frac{\ln(P(0,T)) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) + \frac{KP(0,T)}{S_0}\mathcal{N}\left(\frac{\ln\left(\frac{S_0^2}{K^2P(0,T)}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right).$$

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