Development and Pricing of a New Participating Contract

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Abstract

This article designs and prices a new type of participating life insurance contract. Participating contracts are popular in the United States and European countries. They present many different covenants and depend on national regulations. In the present article we design a new type of participating contract very similar to the one considered in other studies, but with the guaranteed rate matching the return of a government bond. We prove that this new type of contract can be valued in closed form when interest rates are stochastic and when the company can default.

1. Introduction

Life insurance contracts, especially participating contracts, are the subject of a huge literature. Because these contracts bear many covenants (e.g., surrender options, bonus options) and are subject to many potential risks (e.g., interest rates, default, legal and mortality risks), building valuation tools and methods to price and manage them is important. The rise of new accounting standards, with the implementation by the International Accounting Standard Board (IASB) of International Accounting Standards (IAS) 32 and 39 and of International Financial Reporting Standard (IFRS) 4, suggests that fair value has become the key concept of corporate finance and insurance theory. In that sense this evolution calls for a stronger integration of financial and actuarial methods.

Briys and de Varenne (1994, 1997a) built a simple framework for the pricing of life insurance contracts that has been used often in the literature (see, e.g., Grosen and Jørgensen 2002). They value life insurance contracts in a stochastic interest rate environment and take into account the default risk of the issuing company. Nevertheless, as in the Merton model of a firm's capital structure, default here can occur only at the contract maturity. This is an obvious limitation of their framework. Briys and de Varenne (1997b) in a different manner valued risky debt in a stochastic interest rate environment where the default barrier is stochastic, say, proportional to a zero-coupon bond. This model, which is an extension of the Black and Cox model where default can occur at any time, is particularly interesting from a financial viewpoint. To the best of our knowledge, this approach has not yet been used in the insurance domain.

Since the articles of Briys and de Varenne, many papers have been written on participating contracts. Among these contributions Miltersen and Persson (2003) should be noted, who provide closed-form formulas for guaranteed investment contracts with participating covenants. Bacinello (2001) values life insurance contracts with binomial trees. This method proves to be very useful in dealing with various actuarial features: for example, mortality, surrender options, minimum guarantees, annual participating bonuses, and periodic premia. Tanskanen and Lukkarinen (2003) studied participating life insur-

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ance contracts in a Black and Scholes framework. In that case the insured has the right to surrender at some prespecified dates and is able to change the reference portfolio. The pricing is done via numerical schemes. One can also note Ballotta (2005), who evaluated with-profit contracts based on an asset modeled by a jump diffusion process. In conclusion, a huge diversity of products, models, and numerical methods exist with which to tackle the challenge of valuing fair value life insurance contracts.

In Bernard, Le Courtois, and Quittard-Pinon (2005), some participating contracts are priced under both default risk and a stochastic term structure of interest rates. These authors represent the firm assets by a standard geometric Brownian motion correlated with the driving factor of the interest rates. Their interest rate framework is the Vasicek one, within which an exponential volatility structure is used. To solve their problem and avoid heavy numerical schemes, they rely on the extended Fortet (1943) method first developed in finance by Collin-Dufresne and Goldstein (2001), which is based on the implementation of recurrence equations. It should be noted that although use of the extended Fortet's equations allows the latter authors to price participating contracts quite quickly (up to a few minutes of computation time) compared to Monte Carlo methods, their numerical scheme is not instantaneous yet.

In this article we construct a new type of contract close to the one presented in Bernard, Le Courtois, and Quittard-Pinon (2005) but where the minimum guaranteed rate is of a different nature. We assume that, under this new contract, the life insurance company guarantees a government rate to the insured, in other words, a stochastic interest rate: if and when the company defaults, the covenant guarantees a sum proportional to the value of a government bond, whose value is unknown at that time by definition. We shall denote these new contracts as "Government Rate Guarantee Participating Life Insurance Contracts (GP-LICs)." In a similar fashion, the more classical contracts studied by Bernard, Le Courtois, and Quittard-Pinon will be denoted as "Constant Rate Guarantee Participating Life Insurance Contracts (CP-LICs)." The contribution of this article is to show that the introduced new contracts, or GP-LICs, can be priced in closed form under a Vasicek term structure of interest rates. We also do an empirical analysis of this contract's prices and sensitivities.

2. DESIGN OF A NEW CONTRACT

We shall study in this article a particular type of contract (GP-LIC) that is an extension of the participating contract (CP-LIC) studied in Bernard, Le Courtois, and Quittard-Pinon (2005). The originality of this new contract lies in its minimum guarantee, which is proportional to the value of a government bond and hence stochastic.

2.1 Contract Payoff

We classically assume that the funds raised by the insurance company constitute its assets and are modeled by a lognormal diffusion. From the proceeds made by the company, a part is distributed as a minimum rate and another part as a bonus or participating interest on the financial success of the firm. The originality of a GP-LIC lies in its minimum guaranteed rate. We assume that the company guarantees a rate proportional to that of a set of government zero-coupon bonds maturing, as a first step, at the same time T as the contract. Indeed, if the company defaults, the insured recover the amount they would have obtained by investing initially in government bonds, times a proportionality coefficient.

We suppose that the insured invest the initial capital L_0 in the participating policy. Leaving aside the participating bonus temporarily, the minimum guarantee is equivalent to buying $\beta L_0/P(0, T)$ government zero-coupon bonds having an initial value equal to $P(0, T)$ at time 0. Here β is chosen inferior to one, so as to allow the company to guarantee a minimum rate inferior to the government rate. Indeed, at maturity this position is worth $\beta L_0/P(0, T)$ contracts, whose value is equal to $P(T, T) = 1$; in other words, it is worth $l_T^g = \beta L_0/P(0, T)$. Provided $\beta > P(0, T)$, the insured have a minimum amount guaranteed at time T that is superior to their initial investment L_0. Now, if the company defaults at time $t < T$, a minimum amount $l_t^g = \beta L_0 P(t, T)/P(0, T)$ is guaranteed (value at time t of $\beta L_0/P(0, T)$

zero-coupon bonds maturing at time T) by the covenant. This means in particular that we are neglecting—for the time being and at this level—the additional bankruptcy costs that may occur at time t.

Let A be the assets of the firm, modeled as a lognormal process. The initial contribution of the insured, L_0, satisfies the following relationship: $L_0 = \alpha A_0$, where A_0 is the initial value of the assets. For the initial amount of equity, it readily verifies $E_0 = (1 - \alpha)A_0$. The insured receive at maturity T of their contracts, provided the company did not bankrupt in the meantime,

$$\Theta_L(T) = \begin{cases} A_T & \text{if } A_T < l_T^g \\ l_T^g & \text{if } l_T^g \leq A_T \leq \dfrac{l_T^g}{\alpha} \\ l_T^g + \delta(\alpha A_T - l_T^g) & \text{if } A_T > \dfrac{l_T^g}{\alpha}, \end{cases}$$

where, in the first situation, the company has defaulted at maturity, and the assets are returned to the insured; whereas in the second situation, the company performs correctly, and the insured receive the guaranteed rate defined above. Finally, in the third situation, an additional participating rate is redistributed. Note that in the discriminating value l_T^g/α, where the participating rate starts being distributed, the coefficient α appears, putting forward the equitable distribution of benefits between the insured and the equityholders. Initially, the insured possess the amount αA_0, or L_0. At maturity, one simply has to compare αA_T with l_T^g, to decide whether a participation rate is added to the guaranteed rate.

To sum up, the final payoff to the insured can be written as

$$\Theta_L(T) = l_T^g + \delta(\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+, \tag{2.1}$$

where one adds to the sum promised to the insured a bonus option corresponding to a participation in the benefits of the company and a put option directly related to the terminal default risk of the issuing company.

A first idea would be to value these contracts by taking the risk-neutral expectation of the above payoff. This would essentially mean computing a linear combination of standard European options whose closed-form formulas could be easily obtained. In the next paragraph, we shall extend this contract to a setting where default can happen at any time (put differently, we shall move from a Merton-type model to a Black and Cox-type model).

2.2 A More Refined Default Model

A life insurance company must be able to honor its commitments to the insured. It should be solvent at any time, and not only at the maturity of the issued contracts (this corresponds to assuming classically that a covenant forces the company to be solvent at any time even if it pays back the insured only at a fixed date in the future). In the case we consider, the insured has a minimum amount guaranteed l_t^g at any time t if the company defaults early. Let us recall that this is the value at time t of $\beta L_0/P(0, T)$ zero-coupon bonds maturing at time T. Of course, in the real world, this is a basis for the calculation of what is really going to be distributed to the insured upon default, because bankruptcy costs will have to be taken into account.

Default occurs when the level of the assets is not sufficient to reimburse the insured. Let τ be the company's early default time. It may be written as

$$\tau = \inf\{s < T / A_s < \lambda_1 l_s^g\}, \tag{2.2}$$

where λ_1 is a proportionality coefficient whose meaning is that the managers cannot fully anticipate bankruptcy and declare it when the assets attain a level $\lambda_1 l_s^g$ (which is inferior to l_s^g at time s). Now, let λ_2 be the bankruptcy costs parameter: upon default, a fraction of the assets are wasted to cover various costs such as court fees. The amount of assets remaining at default $\lambda_1 l_\tau^g$ has to be diminished using λ_2 to obtain the residual amount $\lambda_2\lambda_1 l_\tau^g$ that will be redistributed to the insured. In full generality, λ_1 and λ_2 should be estimated from past records on the assets values and recovery rates of defaulting life insurance companies.

The payoff to the insured at default time $\tau < T$, assuming $\lambda_1 < 1$ and $\lambda_2 < 1$, may be written as

$$\Theta_L(\tau) = \lambda_1\lambda_2 l_\tau^g \, \mathbb{1}_{\tau<T}. \tag{2.3}$$

The general pricing formula for our contract then can be established using the above expressions $\Theta_L(T)$ and $\Theta_L(\tau)$, given in equations (2.1) and (2.3). Denoting by r the risk-free interest rate process, one readily has under the risk-neutral measure Q the formula allowing one to compute V_1, the value of a GP-LIC:

$$V_1(0) = \mathbb{E}_Q[(e^{-\int_0^T r_s\,ds}\, \Theta_L(T))\, \mathbb{1}_{\tau\geq T} + (e^{-\int_0^\tau r_s\,ds}\, \Theta_L(\tau))\, \mathbb{1}_{\tau<T}].$$

This equation can be written explicitly as

$$V_1(0) = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\, (l_T^g + \delta(\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+)\;\; \mathbb{1}_{\tau\geq T} + e^{-\int_0^\tau r_s\,ds}\, \lambda_1\lambda_2 l_\tau^g \mathbb{1}_{\tau<T}]. \tag{2.4}$$

To do the valuation of this guarantee, one has to postulate some dynamics for the interest rates and assets, which we will do in the following subsection.

2.3 Assets and Interest Rate Dynamics

We set ourselves in a general framework where we need to know for our study the forward-neutral dynamics of the assets A_t and the zero-coupon bonds $P(t, T)$. We assume that the assets follow a lognormal dynamics correlated to the interest rates, which themselves possess an exponential volatility structure σ_P. The interest rate model considered here is driven by a unique factor, correlated to the one of the assets, as mentioned before.

Setting $\nu > 0$ and $a > 0$, the volatility structure is expressed simply as

$$\sigma_P(t, T) = \frac{\nu}{a}(1 - e^{-a(T-t)}).$$

Under the risk-neutral measure Q, the dynamics of the assets A_t and the zero-coupon bond $P(t, T)$ are expressed as

$$\frac{dA_t}{A_t} = r_t\, dt + \sigma\, dZ^Q(t) \tag{2.5}$$

and

$$\frac{dP(t, T)}{P(t, T)} = r_t\, dt - \sigma_P(t, T)\, dZ_1^Q(t),$$

where $Z^Q(t)$ and $Z_1^Q(t)$ are standard Q-Brownian motions with a correlation coefficient equal to ρ.

Let us now construct a Brownian motion Z_2^Q independent from Z_1^Q, that is, such that $dZ_1^Q \cdot dZ_2^Q = 0$. It is possible to split up Z^Q into the two following components:

$$dZ^Q(t) = \rho\, dZ_1^Q(t) + \sqrt{1-\rho^2}\, dZ_2^Q(t).$$

We have therefore decorrelated the pure interest rate risk from the other sources of risk. The dynamics of the assets given in equation (2.5) now can be reexpressed as

$$\frac{dA_t}{A_t} = r_t\, dt + \sigma(\rho dZ_1^Q(t) + \sqrt{1-\rho^2}\, dZ_2^Q(t)).$$

Recall that the Radon-Nikodym density allowing one to build the forward-neutral measure Q_T is defined by

$$\frac{dQ_T}{dQ} = e^{-\int_0^T \sigma_P(s,T)\, dZ_1^Q(s) - (1/2)\int_0^T \sigma_P^2(s,T)\, ds}.$$

In this case the short-term interest rate dynamics obey the relationship

$$dr_t = a(\theta_t - r_t)dt + \nu\, dZ_1^{Q_T}(t),$$

where $\theta_t = \theta - \nu^2/a^2\,(1 - e^{-a(T-t)})$ and we have defined a new Brownian motion $Z_1^{Q_T}$ satisfying under Q_T the relationship: $dZ_1^{Q_T} = dZ_1^Q + \sigma_P(t, T)\, dt$.

We also define $Z_2^{Q_T}$ such that $Z_1^{Q_T}$ and $Z_2^{Q_T}$ be noncorrelated Q_T-Brownian motions. The dynamics of A_t and $P(t, T)$ under Q_T are written as

$$\frac{dA_t}{A_t} = (r_t - \sigma\rho\sigma_P(t, T))\, dt + \sigma(\rho dZ_1^{Q_T} + \sqrt{1 - \rho^2}\, dZ_2^{Q_T})$$

and

$$\frac{dP(t, T)}{P(t, T)} = (r_t + \sigma_P^2(t, T))\, dt - \sigma_P(t, T)\, dZ_1^{Q_T}.$$

After integrating these two dynamics, one obtains

$$A_t = \frac{A_0}{P(0, t)} \exp\left(\int_0^t (\sigma_P(u, t) + \rho\sigma)\, dZ_1^{Q_T}(u) + \int_0^t \sigma\sqrt{1 - \rho^2}\, dZ_2^{Q_T}(u) + \int_0^t \left(-\sigma_P(u, T)(\sigma_P(u, t) + \rho\sigma) + \frac{\sigma_P^2(u, t) - \sigma^2}{2}\right) du\right)$$

and

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp\left(-\int_0^t (\sigma_P(u, T) - \sigma_P(u, t))\, dZ_1^{Q_T} + \frac{1}{2}\int_0^t (\sigma_P(u, T) - \sigma_P(u, t))^2\, du\right).$$

Finally, note that the following dynamics will also be useful:

$$\frac{A_t}{P(t, T)} = \frac{A_0}{P(0, T)} \exp\left(\int_0^t (\sigma_P(u, T) + \rho\sigma)\, dZ_1^{Q_T}(u) + \int_0^t \sigma\sqrt{1 - \rho^2}\, dZ_2^{Q_T}\,(u) - \frac{1}{2}\int_0^t ((\sigma_P(u, T) + \rho\sigma)^2 + \sigma^2(1 - \rho^2))\, du\right) \quad (2.6)$$

Let us now see the exhibition of the main formulas giving the price of a GP-LIC in the setting just defined.

2.4 Main Formulas

We are now able to give a general valuation formula for our guarantee. Starting from equation (2.4), and moving toward the forward-neutral world, one obtains (see the Appendix for more details)

$$V_1(0) = P(0, T)\, \mathbb{E}_{Q_T}[(l_T^g + \delta(\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+)\, \mathbb{1}_{\tau \geq T} + \lambda_1\lambda_2 l_T^g \mathbb{1}_{\tau < T}]. \quad (2.7)$$

We wish to write the above formula in a simplified form:

$$V_1(0) = P(0, T)[GF + BO - PO + LR], \quad (2.8)$$

and we define for this purpose:

$$\begin{cases} GF = l_T^g\,(1 - E_1) \\ BO = \alpha\delta(E_6 - E_2) - \delta l_T^g\,(E_7 - E_3) \\ PO = l_T^g\,(E_8 - E_4) - E_9 + E_5 \\ LR = \lambda_1\lambda_2 l_T^g E_1, \end{cases}$$

where the fundamental contributions to the contract's value express as follows:

$$E_1 = Q_T[\tau < T]$$

$$\begin{array}{ll|ll}
E_2 &= \mathbb{E}_{Q_T}[A_T 1\!\!1_{A_T > (l_T^g/\alpha)} 1\!\!1_{\tau < T}] & E_6 &= \mathbb{E}_{Q_T}[A_T 1\!\!1_{A_T > (l_T^g/\alpha)}] \\
E_3 &= Q_T\left[A_T > \dfrac{l_T^g}{\alpha}, \tau < T\right] & E_7 &= Q_T\left[A_T > \dfrac{l_T^g}{\alpha}\right] \\
E_4 &= Q_T[A_T < l_T^g, \tau < T] & E_8 &= Q_T[A_T < l_T^g] \\
E_5 &= \mathbb{E}_{Q_T}[A_T 1\!\!1_{A_T < l_T^g} 1\!\!1_{\tau < T}] & E_9 &= \mathbb{E}_{Q_T}[A_T 1\!\!1_{A_T < l_T^g}].
\end{array}$$

The next section will compute explicitly the nine contributions defined above.

3. CONTRACT VALUATION

We start by detailing the mechanics allowing to compute the first subcontract term, E_1.

3.1 Computation of E_1

E_1 is the probability that bankruptcy occurs before T. Using equation (2.2), this is the probability that A_u crosses $\lambda\ \beta L_0/P(0, T)\ P(u, T)$, in other words, that the process $A_u/P(u, T)$ crosses the barrier $\lambda_1\beta$ $L_0/P(0, T)$ before T, or written more explicitly:

$$E_1 = Q_T\left(\inf_{u\in[0,T[}\left(\frac{A_u}{P(u, T)}\right) < \lambda_1 l_T^g\right).$$

Start by noting that equation (2.6) is written as

$$\frac{A_u}{P(u, T)} = \frac{A_0}{P(0, T)}\, e^{N_u-(1/2)\xi(u)},$$

where the differential of N is defined by

$$dN_s = (\sigma_P(s, T) + \rho\sigma)\, dZ_1^{Q_T}(s) + \sigma\sqrt{1-\rho^2}\, dZ_2^{Q_T}(s),$$

and the quadratic variation of N is

$$\xi(u) = <N>_u = \int_0^u [(\sigma_P(s, T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)]\, ds. \tag{3.1}$$

The key of the computation of E_1 is the Dubins-Schwarz theorem (see, e.g., Karatzas and Shreve 1991), which states that there exists a unique Q_T-Brownian motion B such that

$$\forall u \in [0, T],\ N_u = N_0 + B_{\xi(u)}.$$

Using this representation theorem, the searched probability becomes

$$\begin{aligned}
E_1 &= Q_T\left(\inf_{u\in[0,T[}\left(\frac{A_u}{P(u, T)}\right) < \lambda_1 l_T^g\right) \\
&= Q_T\left\{\min_{u\in[0,T]}\left(\frac{A_0}{P(0, T)}\, e^{N_u-(1/2)\xi(u)}\right) < \lambda_1 l_T^g\right\} \\
&= Q_T\left\{\min_{u\in[0,T]}\left(e^{B_{\xi(u)}-(1/2)\xi(u)}\right) < \frac{P(0, T)\lambda_1 l_T^g}{A_0}\right\} \\
&= Q_T\left\{\min_{s\in[0,\xi(T)]}\left(B_s - \frac{1}{2}s\right) < \ln(\lambda_1\beta\alpha)\right\}.
\end{aligned}$$

It appears from this formula that we need to know only the law of the minimum of an arithmetic

Brownian motion to compute E_1 (it can be found, e.g., in Jeanblanc, Yor, and Chesney 2006). One then obtains

$$E_1 = \mathcal{N}\left(\frac{\ln(\lambda_1\alpha\beta) + \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + \frac{1}{\lambda_1\alpha\beta}\,\mathcal{N}\left(\frac{\ln(\lambda_1\alpha\beta) - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right).$$

To simplify notation, we use the auxiliary functions η^+ and η^- defined by

$$\eta^+(x) = \mathcal{N}\left(\frac{\ln(x) + \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right), \qquad \eta^-(x) = \mathcal{N}\left(\frac{\ln(x) - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right),$$

where $\mathcal{N}$ denotes the cumulative standard normal distribution function. In this setting one finally obtains for E_1:

$$E_1 = \eta^+(\lambda_1\alpha\beta) + \frac{1}{\lambda_1\alpha\beta}\,\eta^-(\lambda_1\alpha\beta).$$

We shall now see that the computation of E_2, though different from the one of E_1, follows readily, based on the same tools and principles.

3.2 Computation of E_2

Let us first recall the expression of E_2:

$$\begin{aligned} E_2 &= \mathbb{E}_{Q_T}\left[A_T \mathbb{1}_{\{A_T > l^g_T/\alpha\}} \mathbb{1}_{\tau < T}\right] \\ &= \mathbb{E}_{Q_T}\left[A_T \mathbb{1}_{\{A_T > l^g_T/\alpha\}} \mathbb{1}_{\inf_{u\in[0,T]}(A_u/P(u,T)) < \lambda_1 l^g_T}\right]. \end{aligned}$$

To compute this formula, two steps are in order. First, one has to apply Girsanov's theorem to move from the probability Q_T to a new probability $\tilde{Q}$ allowing us to simplify greatly the above expression (by getting rid of A_T). Then, another application of the Dubins-Schwarz theorem under the new probability $\tilde{Q}$ will help us reach E_2.

Define the following Radon-Nikodym measure:

$$\frac{d\tilde{Q}}{dQ_T} = \exp\left(\int_0^T (\sigma_P(u,T) + \rho\sigma)\, dZ_1^{Q_T}(u) + \int_0^T \sigma\sqrt{1-\rho^2}\, dZ_2^{Q_T} - \frac{1}{2}\xi(T)\right),$$

where $\xi(T)$ is defined as in equation (3.1). Thanks to Girsanov's theorem, it is possible to construct under $\tilde{Q}$ the two standard Brownian motions $\tilde{Z}_1$ and $\tilde{Z}_2$ defined by

$$d\tilde{Z}_1(s) = dZ_1^{Q_T}(s) - \int_0^s (\sigma_P(u,T) + \rho\sigma)\, du$$

and

$$d\tilde{Z}_2(s) = dZ_2^{Q_T}(s) - \int_0^s \sigma\sqrt{1-\rho^2}\, du.$$

Let us express A_T under $\tilde{Q}$:

$$A_T = \frac{A_0}{P(0,T)} \exp\left(\int_0^T (\sigma_P(u,T) + \rho\sigma)\, d\tilde{Z}_1(u) + \int_0^T \sigma\sqrt{1-\rho^2}\, d\tilde{Z}_2 + \frac{1}{2}\xi(T)\right).$$

Then the expression of E_2 becomes

$$E_2 = \frac{A_0}{P(0,T)}\,\tilde{Q}\left(A_T > \frac{l^g_T}{\alpha};\ \inf_{u\in[0,T[}\left(\frac{A_u}{P(u,T)}\right) < \lambda_1 l^g_T\right).$$

Define the martingale H as

$$H_s = \int_0^s (\sigma_P(u, T) + \rho\sigma)\, d\tilde{Z}_1(u) + \int_0^s \sigma\sqrt{1-\rho^2}\, d\tilde{Z}_2.$$

Note that the quadratic variation of H is equal to the one of N; therefore, we denote it by ξ. Due to the Dubins-Schwarz theorem, there exists a $\tilde{Q}$-Brownian motion $\tilde{B}$ that satisfies

$$H_s = \tilde{B}_{\xi(s)}.$$

This allows simplifying E_2 as

$$E_2 = \frac{A_0}{P(0, T)}\,\tilde{Q}\left[\tilde{B}_{\xi(T)} + \frac{1}{2}\,\xi(T) > \ln\left(\frac{l_T^g P(0, T)}{\alpha A_0}\right), \inf_{s\in[0,\xi(T)[}\left(\tilde{B}_s + \frac{1}{2}\,s\right) < \ln\left(\frac{\lambda_1 l_T^g P(0, T)}{A_0}\right)\right].$$

Finally, using the joint law of an arithmetic Brownian motion and its infimum, one obtains

$$E_2 = \frac{A_0\lambda_1\alpha\beta}{P(0, T)}\,\eta^+(\lambda_1^2\alpha^2\beta).$$

Applying the same method, we are able to compute the expressions of E_3, E_4, and E_5.

3.3 Final Results

As far as the five first terms are concerned, they are obtained following the above methodology, and write as

$$\begin{cases} E_1 = \eta^+(\lambda_1\alpha\beta) + \dfrac{1}{\lambda_1\alpha\beta}\,\eta^-(\lambda_1\alpha\beta) \\ E_2 = \dfrac{A_0\lambda_1\alpha\beta}{P(0, T)}\,\eta^+(\lambda_1^2\alpha^2\beta) \\ E_3 = \dfrac{1}{\lambda_1\alpha\beta}\,\eta^-(\lambda_1^2\alpha^2\beta) \\ E_4 = \eta^+(\lambda_1\alpha\beta) + \dfrac{1}{\lambda_1\alpha\beta}\,(\eta^-(\lambda_1\alpha\beta) - \eta^-(\lambda_1^2\alpha\beta)) \\ E_5 = \dfrac{A_0}{P(0, T)}\,[\eta^-(\lambda_1\alpha\beta) + \lambda_1\alpha\beta(\eta^+(\lambda_1\alpha\beta) - \eta^+(\lambda_1^2\alpha\beta))]. \end{cases} \tag{3.2}$$

The four last terms, where τ does not appear explicitly, are expressed readily as simple Gaussian functions:

$$\begin{aligned} E_6 &= \Phi_1\left(\mathrm{M}_T; \sqrt{\mathrm{V}_T}; \frac{l_T^g}{\alpha}\right), \qquad & E_7 &= \mathcal{N}\left(\frac{\mathrm{M}_T - \ln\left(\dfrac{l_T^g}{\alpha}\right)}{\sqrt{\mathrm{V}_T}}\right) \\ E_8 &= \mathcal{N}\left(\frac{\ln(l_T^g) - \mathrm{M}_T}{\sqrt{\mathrm{V}_T}}\right), \qquad & E_9 &= \Phi_2(\mathrm{M}_T; \sqrt{\mathrm{V}_T}; l_T^g), \end{aligned} \tag{3.3}$$

where A_T follows a lognormal law with moments M_T and V_T, and where Φ_1 and Φ_2 are defined by

$$\Phi_1(m; \sigma; a) = \mathbb{E}[e^X 1\!\!1_{e^X > a}] = \exp\left(m + \frac{\sigma^2}{2}\right)\mathcal{N}\left(\frac{m + \sigma^2 - \ln(a)}{\sigma}\right)$$

and

$$\Phi_2(m; \sigma; a) = \mathbb{E}[e^X \mathbb{1}_{e^X < a}] = \exp\left(m + \frac{\sigma^2}{2}\right) \mathcal{N}\left(\frac{\ln(a) - m - \sigma^2}{\sigma}\right)$$

when X is a random variable distributed as $\mathcal{N}(m, \sigma^2)$.

Formulas (3.2) and (3.3) are truly closed-form formulas of the market value of our contract and can be computed instantaneously once all the parameters are specified. In the last section, we shall provide some interesting numerical results obtained using these formulas.

4. NUMERICAL ANALYSIS

In the first part of our numerical analysis, we give some numerical results related to GP-LICs and explain how to set the different parameters to obtain fair-priced contracts. A second part is devoted to the comparison of GP-LICs and CP-LICs (whose characteristics are recalled in subsection 4.3). The only difference between these contracts is their guaranteed part. Indeed, the guaranteed interest rate of a GP-LIC is proportional to the yield of a zero-coupon bond $P(0, T)$, while a CP-LIC guarantees a constant interest rate r_g. We will first describe this former contract, before comparing it to GP-LICs.

4.1 Data

We give below our chosen parameter values. Some of them will change during the numerical study, and we shall state whether we take the following values or not:

A_0	100
α	0.85
a	0.4
ν	0.008
ρ	0.2
σ	0.1
T	10
$P(0, T)$	0.6703
λ_1	0.6
λ_2	0.4

Let us first recall the meaning of the above coefficients. A_0 refers to the initial assets value of the company, and α is a coefficient yielding the part invested by the insured (i.e., $L_0 = \alpha A_0 = 85$). The two parameters a and ν define the volatility σ_P of the instantaneous interest rate process. The correlation coefficient between the assets process A and the instantaneous interest rate process r is ρ. The volatility σ of the assets is set at 10%, which is quite low and is due to the presence of investment grade bonds in the portfolio of the insurance company. T stands for the maturity of the contract; we suppose it is equal to 10 years. $P(0, T)$ is a government zero-coupon bond maturing at T. Finally, λ_1 is the scale factor on the threshold triggering bankruptcy and is set to 60%.

Note that some parameters have not been given yet: the participating bonus δ and the proportional coefficient β that determines the minimum guarantee. They will be specified when needed. Let us now present some numerical results on the fair valuation of GP-LICs and their components.

4.2 The New Contract

First recall how the fairness of a contract can be assessed. One has set the contract parameters in such a way as to establish the equality between the initial policyholders' investment L_0 and the initial market value $V_1(0)$ of their contracts. That is, formally:

$$L_0 = V_1(0).$$

$V_1(0)$ can be computed instantaneously thanks to formulas (2.8), (3.2), and (3.3). Assigning a value to all the parameters except one, we obtain the remaining parameter's value by means of a simple rootsearch algorithm. Note that the participation coefficient is extremely easy to obtain in terms of the other parameters (for this, use eq. [2.8]):

$$\delta = \frac{\dfrac{L_0}{P(0, T)} - GF + PO - LR}{\alpha(E_6 - E_2) - l_T^g(E_7 - E_3)}. \tag{4.1}$$

In such contracts two parameters play a key role: the guaranteed rate and the participation coefficient. We first concentrate on the guaranteed rate, which is proportional to the yield of the zero-coupon bond $P(0, T)$. Indeed, our contract guarantees the amount $\beta L_0/P(0, T)$ at time T.

Now, let us define y_0 according as

$$L_0 e^{y_0 T} = L_0 \frac{\beta}{P(0, T)};$$

this is the yield that can be anticipated at time 0 by the insured, assuming subsequent bankruptcy will not occur. Of course, this is only an indication at time 0 of the true yield of a GP-LIC; note also that in this case the yield at time t is unknown because of the stochastic nature of the guarantee.

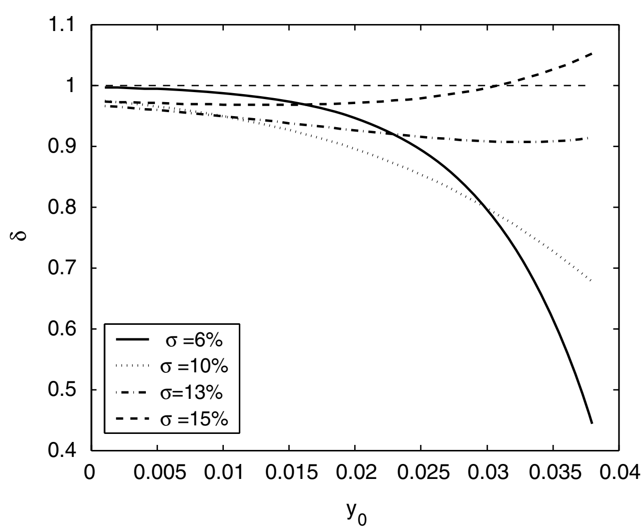
Keeping the values of our list above (except for the volatility σ that is allowed to change), we compute and graph in Figure 1 the fair participation coefficient δ with respect to y_0, at some given fixed values of the volatility ($\sigma = 6\%$, $\sigma = 10\%$, $\sigma = 13\%$, and $\sigma = 15\%$).

The interpretation of the two first curves ($\sigma = 6\%$ and 10%) is very standard: they are negatively sloped because—in a general context—a higher-yield y_0 should be compensated by a lower participation coefficient. When the volatility of the underlying assets portfolio is higher ($\sigma = 13\%$ or $\sigma = 15\%$), one can observe different curves—where δ tends to be an increasing function of y_0. The interpretation is as follows: when σ is high, the insured are facing a quite important default risk; because of this and the fact that extracting more value from the assets (by distributing y_0) increases the ruin probability, policyholders will require higher participating rates δ under higher yields y_0. The horizontal dashed line gives a limit beyond which fair contracts cannot be built without seriously harming shareholders.

Clearly, not every choice of parameters will yield acceptable fair contracts. In particular, some parameters obey regulatory constraints and cannot be fixed arbitrarily. For instance, in France participation coefficients should be higher than 85%, guaranteed interest rates (corresponding here to the

Figure 1

Participation Coefficient δ with Respect to y_0



yield y_0) should be less than 75% of the average governement yield, and α should necessarily be inferior to 96% (existence of minimum solvency margin of 4%). Parameters should range between realistic values: the participation coefficient should preferably be less than 100%, and guaranteed interest rates must of course be positive. These constraints are illustrated in Figure 2, where plots of the participation coefficient δ with respect to α can be observed. In this graph the first coordinate ranges between 80% and 100%, and y_0 is set to 0.5%, 2%, and 3%.

Under the existing constraints some contracts cannot exist; for instance, it is impossible to set $\alpha = 84\%$, $y_0 = 3\%$ and give a participation rate of $\delta = 80\%$. The range of possibilities is located in the grey zone (graphing the constraints $0.85 < \delta < 1$ and $\alpha < 0.96$). An important remark is that when solvency is in danger (when equity decreases or α increases), then for the contract to be fair, its associated participation coefficient should increase.

The above numerical study reveals that the contracts we introduced (GP-LICs) display very standard features. Our next goal is to show in what respects they defer from other existing contracts (CP-LICs).

4.3 Comparison with Existing Contracts

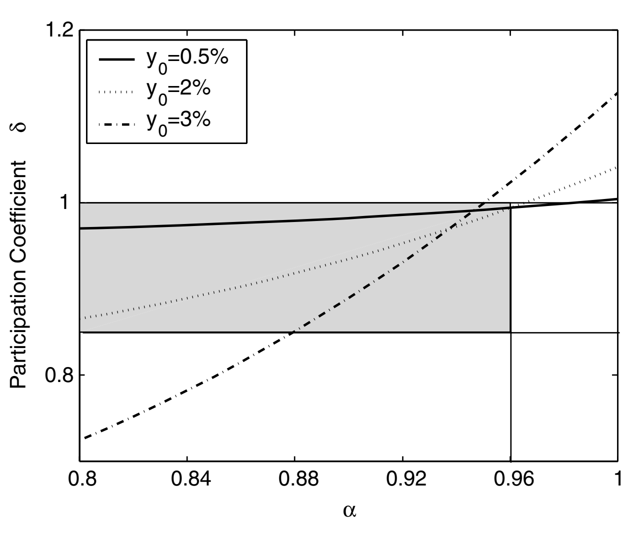
We first recall briefly the design of CP-LICs, based on the description made in Bernard, Le Courtois, and Quittard-Pinon (2005). Then we study the dependence of GP-LICs and CP-LICs on the guaranteed interest rate, the interest rate volatility, and the correlation between the interest rate and the assets processes.

4.3.1 Description

A CP-LIC is a participating contract with minimum and constant (instantaneously compounded) interest rate guaranteed r_g, and a participation coefficient equal to δ. Note that in the case of CP-LICs, r_g holds between 0 and T and is contractual (whereas in the case of GP-LICs, y_0 is just an equivalent yield representing the rate guaranteed from time 0, only, and up to time T).

The initial investment of the insured is L_0; at maturity T, in the case of no prior default, he or she will receive the investment put up by the guaranteed rate, that is, $L_T^g = L_0 e^{r_g T}$. At that time, he or she should also get the participating part of the contract, $\delta(\alpha A_T - l_T^g)^+$, provided the company performed well. Default risk is taken into account by introducing a regulatory barrier in the valuation model of the contract. The level of the assets of the company has to be above $\lambda_1 L_0 e^{r_g t}$ at any given time t; τ is

Figure 2
Participation Coefficient δ with Respect to α



the default or the first passage time of the assets' process at the barrier $B_t = \lambda_1 L_0 e^{r_g t} = \lambda_1 L_t^g$. A CP-LIC therefore admits, under the risk-neutral probability Q, the valuation formula

$$V_2(0) = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\,(L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+)\mathbb{1}_{\tau\geq T} + \lambda_1\lambda_2\, e^{-\int_0^\tau r_s\,ds}\, L_\tau^g \mathbb{1}_{\tau<T}].$$

Under a stochastic interest rate environment, this formula cannot be developed in closed form because the minimum guaranteed interest rate (and hence the default barrier) is deterministic and not proportional to risk-free zero-coupon bonds—as is the case for the new contracts introduced in this article. Instead, this formula can be developed in semiclosed form, as shown by Bernard, Le Courtois, and Quittard-Pinon (2005), and based on the Collin-Dufresne and Goldstein (2001) approach. We use their methodology to price CP-LICs, with a slight adaptation because a clear distinction between λ_1 and λ_2 had not been made in their paper.

In fact, in a simple model where interest rates are constant and not stochastic, this standard contract can be priced in closed form as shown by Grosen and Jørgensen (2002). We will use the notation V_3 for the value of a CP-LIC evaluated in a constant interest rate model. We recall the notation V_1 (respectively V_2) for the price of a GP-LIC (respectively a CP-LIC) evaluated under a stochastic interest rate assumption.

Recall also that the yield of a GP-LIC is proportional to the yield of a government zero-coupon bond maturing at time T. Before comparing the two contracts, we want them to guarantee approximately the same yield. In fact, a GP-LIC's yield can be known only at time 0 (because the contractual guarantee at time t is proportional to $P(t, T)$, which is stochastic). Denoting by r_g the minimum guaranteed rate of a CP-LIC, we set β for the GP-LIC in such a way as to satisfy the following equality:

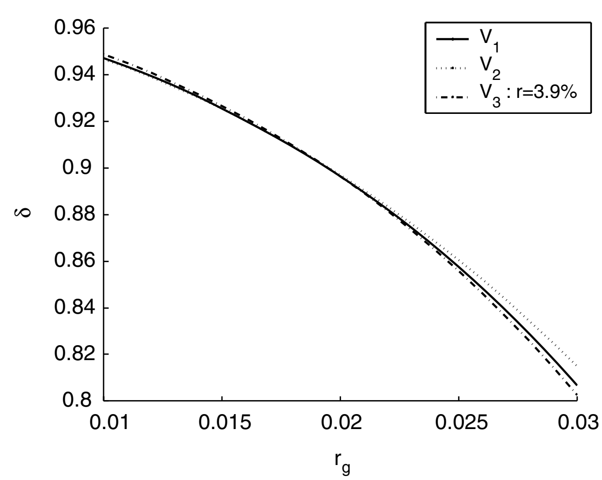
$$L_0 e^{r_g T} = L_0 \beta \frac{1}{P(0, T)},$$

where it is meant that both contracts should start offering the same initial yield (put differently, $r_g = y_0$ should hold).

4.3.2 Participation Coefficient of a Fair Contract

We choose the parameters given earlier and graph in Figure 3 the value of the participation coefficient δ with respect to r_g (where r_g ranges between 1% and 3% and is equal to y_0) for both contracts. Recall that the participation coefficient δ admits a closed-form expression. Indeed, equation (4.1) gives the

Figure 3
δ as a Function of r_g



corresponding formula for the case of GP-LICs (a similar expression can be found in Bernard, Le Courtois, and Quittard-Pinon (2005) for the case of CP-LICs).

We need to affect a value to the constant risk-free interest rate to compute V_3, that is, the value of a CP-LIC in a constant interest rate framework. We chose $r = 3.9\%$ for the pricing of V_3: this is this particular value that makes the third curve in Figure 3 close enough to the two first ones.

Figure 3 is very typical, and its shape can be explained simply: to compensate for a low guaranteed rate, the insurance company has to provide a high level of participation on the assets' performance. In the remainder of this study, we shall assume $r_g = 2\%$, and keep the fair value of δ for each of the existing three situations. This means that we will assume $\delta_1 = 89.70\%$ as far as GP-LICs will be concerned, $\delta_2 = 90.25\%$ for CP-LICs in a stochastic interest rate context, and $\delta_3 = 89.63\%$ for CP-LICs in a constant interest rate context.

4.3.3 Default Probability

We denote by E_1 the default probability, and display its dependence with regards to the minimum guaranteed rate r_g in Figure 4. It is interesting to interpret how the default probability varies. First of all, and this is common sense, it increases with r_g. Indeed, when all the other parameters are kept constant, an increase of r_g means an increase of the payout rate withdrawn from the assets, and hence a higher default probability.

Second and more interesting, for a very analogous design and similar parameters, a GP-LIC is less likely to induce bankruptcy of the issuing company than a CP-LIC. Let us explain this feature. Note first that the CP-LIC's default barrier is defined by

$$\lambda_1 l_T^g e^{-r_g(T-t)},$$

which is the discounted value at r_g of the terminal guaranteed amount.

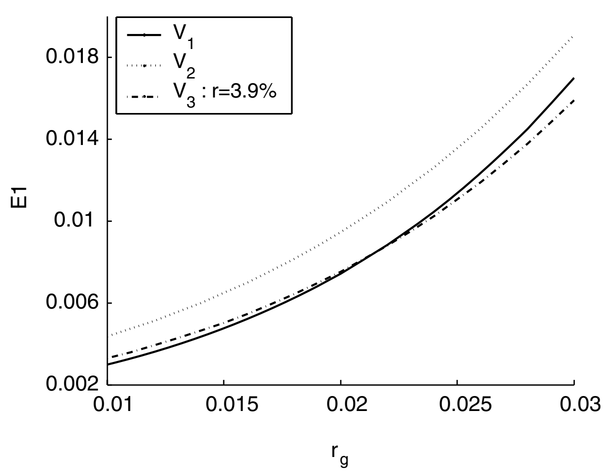
On the other hand, the GP-LIC's default barrier is constructed as

$$\lambda_1 l_T^g P(t, T),$$

which is the terminal guaranteed amount discounted by means of a risk-free zero-coupon bond.

Note that the stochastic interest rate setting constructed above imposes a smaller default barrier than the constant interest rate setting. This is because r_g is usually much smaller than a risk-free zero-coupon bond rate; in other words, $e^{-r_g(T-t)} >> P(t, T)$. Also note that even though $P(t, T)$ is stochastic, in general it will never rise to the level of $e^{-r_g(T-t)}$, because of the very small value of r_g that is postulated.

Figure 4
E_1 with Respect to r_g



The conclusion is that GP-LICs, although built with floating barriers (moving in standard market conditions in the opposite way of the assets), bear smaller default probabilities than CP-LICs. This is because the barrier of GP-LICs is always smaller than that of CP-LICs, because of the regulatory low value of r_g.

Note also that E_1 is small for CP-LICs computed with V_3, in other words, with a constant interest rate model. Of course, these values (that would be obtained as simplifications by actuaries and would lead them to underestimate the risk of CP-LICs) are wrong. A constant interest rate model is not sufficient to price efficiently such contracts (for instance, and obviously, it cannot take into account the correlation between the assets and interest rates), and this is another conclusion of our study.

4.3.4 Sensitivity to the Assets Volatility σ and Interest Rate Parameter ν

Let us recall that the parameter values given earlier. We assumed $r_g = 2\%$ and computed the fair value of δ for each of the existing three situations ($\delta_1 = 89.70\%$, $\delta_2 = 90.25\%$, $\delta_3 = 89.63\%$). There the assets' volatility is set to 10%. So the situation $\sigma = 10\%$ represents a fair contract.

We graph in Figure 5 the contract value as a function of the underlying assets volatility σ. The first element that can be noticed is that all contracts have values that start increasing with the level of volatility and then decrease (after an optimum at $\sigma = 10\%$) when the volatility continues increasing. Actually, the optimum corresponds to the fair coefficients computed based on our parameters. All parameters being equal, we observe the impact of a volatility change on the contract market value. Decreasing the volatility below $\sigma = 10\%$ corresponds to decreasing the appeal of the product to the investors. It necessarily decreases its market value. Increasing it induces an increase of the default probability and therefore a decrease of the policy value. This effect is indeed related to the fact that we started from a fair contract.

Comparing both contracts, it appears that V_1 always remains higher than V_2 and has a smaller tendency to decrease with respect to the volatility σ. Again, this is an advantage of GP-LICs. Less sensitive to the volatility of the assets, it should be of greater interest to the investors. This is, of course, a consequence of the way we construct the floating-rate guarantee (reducing in particular the default probability of the issuing company), which is the only distinction between the two contracts. This type of contract, which is less likely to induce default, is compatible with a higher level of σ, and this is a sufficient reason for it to be worth more than a CP-LIC.

Finally, Figure 6 clearly exhibits the higher sensitivity to interest rates of a CP-LIC compared to a GP-LIC. Here ν is the parameter driving the size of the volatility of the interest rates, and could be

Figure 5
***V* with Respect to σ**

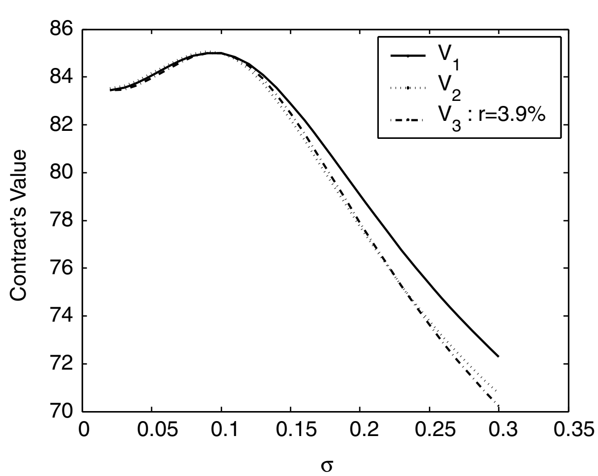
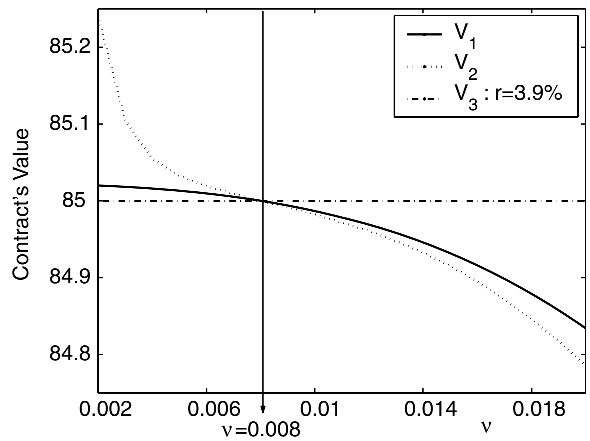


Figure 6
***V* with Respect to ν**



interpreted as a proxy of this volatility itself. Our conclusion at this stage is that a GP-LIC is more akin to resist environment changes than a standard participating contract.

5. Conclusion

In a general framework taking into account actual features such as stochastic interest rates and default probability, we suggest studying a new contract bearing many characteristics of usual participating life insurance contracts. This new contract is designed in a way that leads to an easy understanding of its behavior. Indeed, it reacts to financial and economic parameters qualitatively in a similar way as classical participating contracts and seems to have very interesting management properties. The main technical argument for building this contract is that we value it in closed form. The practical construction of GP-LIC contracts, based on the use of government bonds, which are extremely liquid instruments, seems easy to achieve.

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Appendix

Going Forward-Neutral

We explain briefly how to go from formula (2.4)

$$V(0) = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\,(l_T^g + \delta(\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+)\mathbb{1}_{\tau\geq T} + e^{-\int_0^\tau r_s\,ds}\,\lambda_1\lambda_2 l_T^g \mathbb{1}_{\tau<T}]$$

to formula (2.7)

$$V(0) = P(0, T)\ \mathbb{E}_{Q_T}[(l_T^g + \delta(\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+)\mathbb{1}_{\tau\geq T} + \lambda_1\lambda_2 l_T^g \mathbb{1}_{\tau<T}].$$

The main difficulty here is to show that

$$\mathbb{E}_Q[e^{-\int_0^\tau r_s\,ds}\;\lambda_1\lambda_2 l_T^g \mathbb{1}_{\tau<T}] = P(0, T)\mathbb{E}_{Q_T}[\lambda_1\lambda_2 l_T^g \mathbb{1}_{\tau<T}]$$

because the passage from the risk-neutral probability to the forward-neutral one is direct in the first part of formula (2.4) and simply stems from the definition of these two worlds.

One can write

$$\mathbb{E}_Q[e^{-\int_0^\tau r_s\,ds}\;\lambda_1\lambda_2 l_T^g \mathbb{1}_{\tau<T}] = \lambda_1\lambda_2 l_T^g \mathbb{E}_Q[e^{-\int_0^\tau r_s\,ds}\;P(\tau, T)\,\mathbb{1}_{\tau<T}],$$

where we are discounting a payoff of $P(t, T)\ \mathbb{1}_{\tau<T}$ from τ to 0.

Taking as new numéraire $P(., T)$, one can write under Q_T:

$$\mathbb{E}_Q[e^{-\int_0^\tau r_s\,ds}\;\lambda_1\lambda_2 l_T^g \mathbb{1}_{\tau<T}] = \lambda_1\lambda_2 l_T^g P(0, T)\mathbb{E}_{Q_T}\left[\frac{P(\tau, T)\,\mathbb{1}_{\tau<T}}{P(\tau, T)}\right],$$

which immediately simplifies as

$$\mathbb{E}_Q[e^{-\int_0^\tau r_s\,ds}\;\lambda_1\lambda_2 l_T^g \mathbb{1}_{\tau<T}] = \lambda_1\lambda_2 l_T^g P(0, T)\mathbb{E}_{Q_T}[\mathbb{1}_{\tau<T}],$$

and then the result follows.

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