Optimal Reinsurance Arrangements  
Under Tail Risk Measures  

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Keywords: Optimal Reinsurance, Risk Measures, Alternative Risk Transfer

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Abstract

Regulatory authorities demand insurance companies to control their risk exposure by imposing stringent risk management policies. This article investigates the optimal risk management strategy of an insurance company subject to regulatory constraints. We provide optimal reinsurance contracts under different tail risk measures and analyze the impact of regulators’ requirements on risk-sharing in the reinsurance market. Our results underpin adverse incentives for the insurer when compulsory Value-at-Risk risk management requirements are imposed. But economic effects may vary when regulatory constraints involve other risk measures. Finally, we compare the obtained optimal designs to existing reinsurance contracts and alternative risk transfer mechanisms on the capital market.

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Introduction

European insurance companies have recently experienced increasing stress to incorporate the strict new risk management rules and new capital requirements set by Solvency II. One key component of the Solvency II project is to determine the economic capital based on the risk of each liability in order to control the probability of bankruptcy (equivalently the Value-at-Risk\(^1\)). This paper examines the optimal risk management strategy when this type of compulsory regulatory constraint is imposed to the insurer as well as alternative risk constraints. Moreover, this paper compares the optimal reinsurance policies with existing reinsurance contracts and financial derivatives in the capital market.

We obtain the optimal risk management policies when popular risk measures are implemented. We show that if the insurer minimizes the insolvency risk while limiting the premium spent for reinsurance, an optimal strategy is to purchase a reinsurance contract to insure medium losses but not large losses. Therefore, the Value-at-Risk could induce adverse incentives of the insurer not to buy insurance against large losses. The same strategy is optimal when the insurer wants to minimize the Conditional Tail Expectation of the loss\(^2\). However, the optimal reinsurance contract is a deductible when the insurer minimizes the expected variance. Hence, the optimal risk management policy varies in the presence of different risk measures.

There are several important implications of our results. First, our results confirm that regulation may induce risk averse behaviors of insurers and increase the reinsurance demand. Mayers and Smith (1982) were the first to recognize that insurance purchases are part of firm’s financing decision. The findings in Mayers and Smith (1982) have been empirically supported or extended in the literature. For example, Yamori (1999) empirically observes that Japanese corporations can have a low default probability and a high demand for insurance. Davidson, Cross and Thornton (1992) show that the corporate purchase of insurance lies in the bondholder’s priority rule. Hoyt and Khang (2000) argue that corporate insurance purchases are driven by agency conflicts, tax incentives, bankruptcy costs and regulatory constraints. Hau (2006) shows that liquidity is important for property insurance demand. In a series of research papers, Froot, Sharfstein and Stein (1993) and Froot and Stein (1998) explain the firm would behave risk averse because of the risk management policies that are used to address the presence of costs. This paper contributes to the extensive literature aiming to explain why risk neutral corporations purchase insurance. As shown in this paper risk neutral insurers may behave as risk averse agents in the presence of regulations.

Second, our results provide some rationale of the conventional reinsurance contracts and link existing reinsurance contracts with derivative contracts available in the capital market. Froot (2001) observed that “most insurers purchase relatively little cat reinsurance against large events”.

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\(^1\)For the current stage of Solvency II we refer to extensive documents in http://www.solvency-2.com.

\(^2\)The idea of the Conditional Tail Expectation (CTE) is to capture not only the probability to incur a high loss but also its magnitude. From a theoretical perspective CTE is better than VaR (see Artzner et al. (1999), Inui and Kijima (2005)), and it has already been implemented to regulate some insurance products (with financial guarantees) in Canada (Hardy (2003)).
In actual reinsurance contracts, the insurer often chooses to insure losses above a retention level up to a limit. Using a theoretical approach, Froot (2001) shows that the “excess-of-loss layers” are however suboptimal and that the expected utility theory can not justify the capped features of the reinsurance contracts in the real world. Froot (2001) provides several reasons for these departures from the theory. Our paper also partially justifies the existence of “excess-of-loss layers” from a different angle. When the insurer implements risk management strategies based either on the VaR or the CTE, the insurer is not willing to hedge large losses but purchases insurance only against medium losses. As a consequence, the optimal risk management strategy involves insuring medium losses more than large losses which is consistent with the empirical evidence of Froot (2001). However, we explain later in the paper than the optimal policies show discontinuities and could not be sold in the presence of asymmetric information and moral hazard. Thus our model only partially supports the empirical findings of Froot (2001).

Third, our results offer some risk-sharing analysis in the reinsurance market. This analysis and the methodology could be helpful for both insurers and insurance regulators to compare the effects of imposing risk constraints on insurers. These results also help insurance regulatory to investigate which risk measure is appropriate.

The organization of this paper is as follows. In the next section, a theoretical framework is introduced, and we derive the optimal reinsurance contract under the VaR risk measure. In the “Optimal Reinsurance Arrangements under CTE and other Risk Measures” section we solve the optimal reinsurance design problem under other tail risk measures. We show that the optimal coverage under CTE measure is similar to the optimal coverage under VaR measure. However, under an alternative risk measure, a deductible contract is optimal. Then we compare the optimal reinsurance design with previous literature when the firm behaves risk averse in other frameworks. Our comparison is presented in “Optimal Indemnity with Financing Imperfections”. In the “Reinsurance Market, Capital Market” section we compare the optimal reinsurance contracts under risk measures to contracts frequently sold by reinsurers in the marketplace.

1 Optimal Reinsurance Design under VaR Measure

Consider a one-period reinsurance market model. At the initial time, the insurer receives premia from its customers, and in exchange to these premia, it is obligated to provide coverage at the end of the period. The aggregate amount of indemnities paid in the future is denoted by $X$. Its initial wealth $W_0$ is composed of the collected premia and its own capital. Its final wealth, at the end of the period, is $\hat{W} = W_0 - X$ if no reinsurance is purchased. Since our main purpose is to examine the effects of risk measures on the insurer, the insurer is assumed to be risk-neutral.

As some risks cannot be diversified (e.g. longevity risk, catastrophic risk), we suppose the insurer faces a risk of large loss. We also assume that regulators require the insurer to meet some risk

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3One justification of the risk-neutral assumption is that many insurers hold a well-diversified portfolio.
management requirement. As an example, assume that $\nu$ is a Value-at-Risk (VaR) limit to the confidence level $\alpha$, both parameters $\nu$ and $\alpha$ are often suggested by regulators, then the VaR requirement for the insurer is

$$P \left\{ W_0 - \hat{W} > \nu \right\} \leq \alpha. \quad (1)$$

This probability $P \{ W_0 - \hat{W} > \nu \}$ measures the insolvency risk. This type of risk management constraint has been described explicitly in Solvency II.

Assume there is a competitive reinsurance market and the insurer purchases a reinsurance contract from a reinsurer, paying an initial premium $P$. When $X$ is observed, an indemnity $I(X)$ is transferred from the reinsurer to the insurer. Then, the insurance company’s final wealth becomes $W = W_0 - P - X + I(X)$. The indemnity $I(X)$ is understood as a function of the loss variable $X$. Following classical insurance literature (see for example Arrow (1971) and Raviv (1979)), the coverage $I(X)$ is non-negative and can not exceed the size of the loss ($0 \leq I(X) \leq X$).

To design a reinsurance agreement, we assume the following premium principle $P$.

$$P = \mathbb{E}[I(X) + C(I(X))] \quad (2)$$

where the cost function $C(.)$ is non-negative and satisfies $C''(.) > -1$. Note that this assumption is fairly general and include many premium principles as special cases. For instance, Arrow (1963) shows the deductible policy is optimal when the premium depends on the expected payoff of the policy only. Raviv (1979) extends Arrow’s analysis to the convex cost structure. Huberman, Mayers and Smith (1983) introduce concave cost structure and find deductible might not be optimal.

The final loss $L$ of the insurance company is $L = W_0 - W = P + X - I(X)$, a sum of the premium $P$ and the retention of the loss $X - I(X)$. The VaR requirement is formulated as $P \{ L > \nu \} \leq \alpha$. It is equivalent to $\text{VaR}_L(\alpha) \leq \nu$. A simple reflection implies that the premium must be smaller than the VaR limit $\nu$. Otherwise, say $P > \nu$, then the loss $L = P + X - I(X) > \nu$ because $I(X)$ is non-negative. Hence, the VaR requirement could not be satisfied if the insurance company pays a premium $P$ which is greater than the VaR limit $\nu$.

Our subsequent discussion of the optimal risk management policy is related to an optimal reinsurance design following classical expected utility literature initiated by Arrow (1963, 1971), Borch (1971), and Raviv (1979). In this approach one often finds the optimal (re)insurance contract subject to a limited (re)insurance premium. Precisely, we study the following Problem [1,4].

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[1] VaR is defined by $\text{VaR}_L(\alpha) = \inf \{ x, \ P \{ L > x \} \leq \alpha \}$.

Problem 1.1 Find a reinsurance contract \( I(X) \) that minimizes insolvency risk:

\[
\min_{I(X)} \mathbb{P}\{W < W_0 - \nu\} \quad \text{s.t.} \quad \begin{cases} 
0 \leq I(X) \leq X \\
E[I(X) + C(I(X))] \leq \Delta
\end{cases}
\]

Problem 1.1 solves the minimum insolvency risk, or equivalently, minimizes the probability \( \mathbb{P}(L > \nu) \) that the loss \( L \) exceeds the VaR limit, by paying at most a premium \( \Delta \) to purchase the reinsurance contract. The probability \( \mathbb{P}\{W < W_0 - \nu\} \) can be written as an expected utility \( E[u(W)] \) with a utility function \( u(z) = 1_{z < W_0 - \nu} \). This utility function, however, is not concave. Therefore, standard Arrow-Raviv first-order conditions are not sufficient to characterize the optimum. Note that this remark also applies to subsequent problems investigated in the paper.

Before proceeding, we add one comment on the concept of optimality discussed in this paper. Throughout this paper, we focus on the optimal shape of the contract in a Pareto-optimality framework while the premium principle \( \mathcal{P} \) is imposed. The optimal premium level, which is not addressed in this paper, can be solved via a numerical search (See Schlesinger (1981), Meyer and Ormiston (1999)).

Problem 1.1 can be motivated as follows. Note that \( E[W] = W_0 - E[X] - P + E[I(X)] = W_0 - E[X] - E[I(X) + C(I(X))] + E[I(X)] \). Therefore,

\[
E[I(X) + C(I(X))] \leq \Delta \iff E[W] \geq W_0 - E[X] - \Delta + E[I(X)].
\]

If the risk is captured by estimating the probability of insolvency, then Problem 1.1 characterizes the efficient risk-return profile between the minimal guaranteed expected wealth and the insolvency risk measured by the probability that losses exceed the VaR limit. This point will become clear in Figure 2 below. The problem of minimizing the probability of falling below a minimum level has been studied in the portfolio optimization literature. This problem shares some similarities with the Roy (1952)’s safety-first optimal portfolio. The “safety first” criterion is a risk management technique that allows you to select one portfolio over another based on the criteria that the probability of the return of the portfolios falling below a minimum desired threshold is minimized. Roy (1952) also obtains the efficient frontier between risk and return measured respectively by the default probability and the expected return. Later Figure 2 also displays this efficient frontier in our model.

Problem 1.1 can also be well motivated by investigating its dual problem. Since Problem 1.1 examines the minimum default probability by imposing a upper bound on the premium, its dual problem is clearly to find the minimal possible premium such that the insolvency probability is bounded by the confidence level \( \alpha \). An insurer can find the optimal reinsurance contract either based on Problem 1.1 or its dual problem, according to whether the premium or the confidence level is emphasized. The optimal reinsurance contracts of both problems are similar.

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\(^{6}\)In the literature, there are two separate concepts, one is to determine the optimal shape of the insurance contract and one is to find the optimal level of insurance.
Problem 1.1 optimizes the insurer’s objective under the VaR risk management constraint. But it is only one way to illustrate the insurer’s motivation to search for the optimal risk management strategy. In Problem 1.1 we ignore the interests of the debtholders (policyholders and bondholders) of the insurer. Even a small probability of default might lead to a huge loss of the debtholders. Therefore, the objective function in Problem 1.1 is not necessarily optimal to debtholders. We will come back to this issue after presenting the precise solution of Problem 1.1. Instead of examining a situation in which shareholders, debtholders and regulators are all participating, our objective in Problem 1.1 is more modest and focus on the insolvency probability only.

Moreover, the agency problem between the shareholder and the managers is not addressed in Problem 1.1. As stated earlier, Solvency II does force the insurer to minimize the probability of bankruptcy. The interests between the managers and the shareholders, however, might be different. On one hand, the objective in Problem 1.1 can be understood as an optimal strategy for the managers as the unemployment risk naturally follows from default risk. On the other hand, a more natural optimization problem, from the shareholder’s perspective, is to maximize the expected wealth subject to a similar probability constraint which is the dual problem of Problem 1.1 or subject to a limited liability constraint (Gollier, Koehl and Rochet (1997)). The optimal risk management problem in this context becomes more challenging than Problem 1.1. For instance, Gollier, Koehl and Rochet (1997) find that optimal exposure to risk of the limited liability firm is more risk-taking than the firm under full liability. This point is also exemplified by Ross (2004) in another context, that a concave function shifted by a convex transformation is not necessarily a concave function anymore. The problem of the shareholder is even harder if a probability constraint is imposed in this expected utility framework. Because we confine ourselves to a risk-neutral framework, the discussion of those extended optimal reinsurance problems is beyond the scope of this paper.

Let \( S := \{ P : 0 \leq P < E \left[ (X - \nu + P)^+ + C \left( (X - \nu + P)^+ \right) \right] \} \). The solution of Problem 1.1 is given in the following Proposition.

**Proposition 1.1** Assuming \( X \) has a continuous cumulative distribution function. Assume that \( P \in S \). Let

\[
I_P(X) = (X + P - \nu)^+ 1_{\nu - P \leq X \leq \nu - P + \kappa_P},
\]

where \( \kappa_P > 0 \) satisfies \( E [I_P(X) + C(I_P(X))] = P \). Put

\[
L(P) = P \{ X > \nu - P + \kappa_P \}, \quad \forall P \in S.
\]

---

7 The emerging large default risk often leads to loss of confidence of the managers, significant drops of the share price, pressure from directors and shareholders etc. All these factors make managers worry about their employment status.

8 We refer to Basak and Shapiro (2001), Boyle and Tian (2007) for a similar problem in finance.

9 We allow the case when there is a mass point at 0 meaning \( P(X = 0) \) can be positive.
Then $I_{P^*}(X)$ is an optimal reinsurance contract of Problem 1.1 where $P^*$ solves the static minimization problem

$$\min_{0 \leq P \leq \Delta} L(P).$$

(5)

The proof of Proposition 1.1 is given in Appendix A. According to Proposition 1.1, the optimal reinsurance coverage of Problem A.1 involves a deductible for small losses, and no insurance for large losses. We call it “truncated deductible”. The deductible level in this truncated deductible is the VaR limit minus the premium, and a limit of the insured amount is $\nu - P + \kappa P$. The solution of Problem A.1 involved two stages. In the first stage, we need to find a truncated deductible with indemnity $I_{P}(X)$ by solving the premium equation: $E[I_{P}(X) + C(I_{P}(X))] = P$. Since both the VaR limit $\nu$ and $P$ is fixed in this stage, only one variable $\kappa P$ is solved in this premium equation. Hence the relationship $P \to \kappa P$ is characterized in this step. The amount $\nu - P + \kappa P$ is the upper limit of insurance or maximum loss insured in the reinsurance contract $I_{P}(X)$, the probability $L(P)$ is the probability of the loss exceeding the limit $\nu - P + \kappa P$. Consequently, the second stage is to find the premium $P^*$ with the smallest probability $L(P)$.

Even though the proof of Proposition 1.1 is fairly complicated, this proposition is intuitive. Recall that the objective in Problem 1.1 is to minimize the insolvency probability. The optimal contract must be one in which the indemnity on “bad” states are transformed to the “good” states by keeping the premium (total expectation) fixed. Because reinsurance is costly, it is optimal not to purchase reinsurance for the large loss states.

An important consequence of the non-concavity of the objective function is that the premium constraint $E[I(X) + C(I(X))] \leq \Delta$ is not necessarily binding. In other words, the probability $L(P)$ is not necessarily monotone, hence, the optimal premium $P^*$ does not necessarily satisfy the premium equation $E[I(X) + C(I(X))] = \Delta$. This particular feature of the optimal solution of Problem 1.1 is a surprise because, intuitively, the higher premium paid should lead to smaller solvency probability. This property follows from the specific shape of the truncated deductible indemnity. When the premium $P$ increases, the deductible $\nu - P$ decreases, but the limit $\nu - P + \kappa P$ could increase or decrease. As will be proved in Appendix A, $I_{P}(X)$ is an optimal reinsurance contract with the same premium $P$, and $L(P) = P\{X > \nu - P + \kappa P\}$, the probability $L(P)$ does not decrease in general.

As an example to illustrate this property, we consider a numerical example in which $W_0 = 1000$, $\Delta = \nu = \frac{W_0}{10} = 100$, the loss $X$ is distributed with the following density function (triangle shape):

$$f(x) = \frac{2}{ab} x 1_{x \leq a} + \left( \frac{2}{b} + \frac{2}{b(b-a)}(a-x) \right) 1_{a < x \leq b}.$$
where \( a = 25 \) and \( b = 250 \). The premium principle is \( P = (1 + \rho)E[I(X)] \) where \( \rho = 0.12 \) is the loading factor. Then for each value of \( P \in (0, \nu) \), we solve the quantile \( q \) so that:

\[
\frac{P}{1 + \rho} = E \left[ (X - (\nu - P))^+ \mathbf{1}_{X \leq q} \right]
\]

We then compute the probability of \( X \) to be more than \( q \), which is \( L(P) \). Figure 1 displayed a U-shape of the probability function \( L(P) \): \( L(P) \) is first decreasing and then increasing with respect to the premium \( P \). Clearly, the optimum \( \min_{0 \leq P \leq \Delta} L(P) \) is not the same as the premium \( \Delta \). In fact, the minimum is attained when the optimal premium \( P^* \) is approximately equal to 74 while \( \Delta = 100 \) (see Figure 1).

![Figure 1: Probability L(P) w.r.t. P](image)

We now see why the regulator’s risk measure constraint induces risk aversion to risk neutral insurers. In our model it is optimal for a risk neutral insurer not to buy insurance in the absence of regulators (since they are risk-neutral and that insurance is costly). When there exists a regulation constraint, the optimal insurance design is derived as a truncated deductible. Moreover, the insurance demand can be measured by the amount spent on insurance, which is the premium \( P \). Figure 2 displays the trade-off between the expected return (through the final expected wealth) with respect to the risk (measured by the confidence level \( \alpha \)). Figure 2 shows that this trade-off shape is concave which is similar to the trade-off between risk and return for a risk-averse investor. This figure clearly implies that risk neutral insurance companies react as risk-averse investors in the presence of regulators (this is an induced risk aversion).10

There are other factors which motivate the risk aversion of risk neutral insurers. We will provide a comparison in Section E of our approach with the previous contributions of Froot et al. (1993) and Caillaud et al. (2000).

10Terminology proposed by Caillaud, Dionne and Jullien (2000) in another context.
Figure 2: Expected Wealth w.r.t. the probability to exceed the VaR limit

Assume $X = e^Z$ where $Z$ is a Gaussian random variable $\mathcal{N}(m, \sigma^2)$ where $m = 10.4$ and $\sigma = 1.1$. $W_0 = 100,000$, $\rho = 0.15$. In this figure, the maximum risk and maximum expected final wealth are obtained when no insurance is purchased.

If the VaR limit $\nu$ is set to be the initial wealth $W_0$, Problem 1.1 is interpreted in terms of ruin probability:

$$\min_{I(X)} \mathbb{P}\{W < 0\} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ \mathbb{E}[I(X)] + C(I(X))] \leq \Delta \end{cases}$$

In this special case, the problem is considered by Gajek and Zagrodny (2004). Even though they discover the same optimal shape, their construction of the optimal coverage is not explicit. The same minimal ruin probability problem is also studied in Kalusza and Okolewski (2008) under a different premium principle.

To conclude the discussion of this section, we refer to an optimal reinsurance problem recently studied by Wang et. al (2005). In Wang et. al (2005), the optimal reinsurance contract is the one that maximizes the expected wealth subject to the probability constraint $\mathbb{P}\{W > \mathbb{E}[W] - \nu\} \geq 1 - \alpha$, where the premium is $(1 + \rho)\mathbb{E}[I(X)]$. At first sight, this problem seems to be the same as the dual problem of Problem 1.1. In fact, there exists some significant differences in its economic insight. Note that

$$W - \mathbb{E}[W] = W_0 - X + I - (1 + \rho)\mathbb{E}[I(X)] - \mathbb{E}[W_0 - X + I - (1 + \rho)\mathbb{E}[I(X)]$$

$$= (I(X) - \mathbb{E}[I(X)]) - (X - \mathbb{E}[X]).$$

The probability requirement $\mathbb{P}\{W > \mathbb{E}[W] - \nu\} \geq 1 - \alpha$ is independent of the premium $\Delta$. As a consequence, the minimal insolvency probability considered in Wang et. al (2005) is independent
of the premium of the indemnity contract. Clearly this is not consistent with the conventional VaR risk management practice.\footnote{Moreover, there exists other technical issues in Wang et. al (2005). Our method in proving Proposition\ref{prop:1.1} can be applied to their problem. More details can be obtained from authors upon request.}

\section{Optimal Reinsurance under Tail Risk Measures}

We have explored the optimal reinsurance contract under VaR constraint. Adopting an optimal reinsurance arrangement under VaR constraint, insurance companies choose to leave the worst states uninsured. As has been explained in the last section, this property follows from the fact that worst states are the most expensive ones to insure against. Indeed, such contracts reduce the probability to incur large loss but they do not limit the losses’ amount in worst states. Thus VaR requirements lead companies to ignore losses in the right tail of the loss distribution. A deeper analysis of the consequences of Value-at-Risk management in the financial market can be found in Basak and Shapiro (2001). They show that VaR risk managers often choose larger risk exposure to risky assets and consequently incur larger losses when losses occur.

In this section we discuss optimal reinsurance when other risk measures are imposed. For instance, the Conditional Tail Expectation (CTE) risk measure has been studied extensively in theory. Nowadays, it is used by Canadian regulation in practice (for some index linked insurance annuities). The motivation of CTE is to limit the risk exposure toward large losses instead of the probability of bad states only. We show that, despite of its theoretical advantage (see Artzner et al. (1999), and Basak and Shapiro (2001)), the CTE constraint implies the same optimal design as in the last section. Then we consider another risk measure, which is linked to the second moment of the excess of loss. We prove that the deductible policy is optimal when the second risk measure constraint is satisfied.

For simplicity of notations we assume a linear premium principle \( P = (1 + \rho)E[I(X)] \), where \( \rho \) is a constant loading factor. The results of this section can be easily adapted to the premium that is a function of the actuarial value of the indemnity.

\subsection{CTE Risk Measure}

The optimal design under the conditional tail expectation can be stated as follows.

\textbf{Problem 2.1} Solve the indemnity such that

\[
\min_{I(X)} \{ E \left[ (W_0 - W) I_{W_0 - W > \nu} \right] \} \quad \text{s.t.} \quad \begin{cases} 0 \leq I(X) \leq X \\ (1 + \rho)E[I(X)] \leq \Delta \end{cases}
\] (6)
where the loss level $\nu$ is exogenously specified. We point out that this framework is consistent with the one investigated in Basak and Shapiro (2001), where $\nu$ is not related to $(1-\alpha)$-quantile of the loss. For the linear premium principle, Problem 2.1 solves the optimal reinsurance contract with minimum loss exposure on the loss states $\{W_0 - W > \nu\}$ by paying at most premium $\Delta$.

We state the solution of Problem 2.1 as follows.

**Proposition 2.1** Assuming $X$ has a continuous cumulative distribution function strictly increasing on $[0, +\infty)$. Assume that $P \in S$, Let

$$I_P^c(X) = (X + P - \nu)^+1_{\nu-P \leq X \leq \nu-P+\lambda_P}$$

where $\lambda_P > 0$ satisfies that $(1 + \rho)E[I_P^c(X)] = P$. Let $W_P^c$ be the final wealth derived from $I_P^c(X)$. Then the indemnity $I_P^c(X)$ solves Problem 2.1 when $\Delta^*_c$ minimizes:

$$\min_{0 \leq P \leq \Delta^*_c} E \left[ (W_0 - W_P^c)1_{W_0-W_P^c > \nu} \right]$$

The proof of Proposition 2.1 is contained in Appendix A. Note that $I_P^c(X)$ is the same as $I_P(X)$ of Proposition 1.1 for a linear premium principle. Then $\lambda_P = \kappa_P$. The optimal $P^*$ in both Proposition 1.1 and 2.1 are different because of different objective functions in the second stage of the solution. Hence the script “c” is still used to denote the CTE constraint for short.

Proposition 2.1 shows that a risk neutral insurer behaves risk averse in the presence of CTE risk measure constraint since it is optimal to buy some reinsurance. Moreover, insurers have no incentives to protect themselves against large losses under the conditional tail expectation’s constraint. This proposition derives the same adverse incentive for the insurer as in the VaR case. This result seems surprising because people often argue that CTE risk measure works better than the VaR risk measure (see Basak and Shapiro (2001), Artzner et al. (1999)). In fact, this result is still intuitive. Recall $L = W_0 - W$ is the loss. In terms of the loss variable $L$, Problem 2.1 is to solve for

$$\min_L E[L1_{L > \nu}]$$

subject to $P \leq L \leq P + X$. Therefore, the objective is to take consideration of the trade-off between the amount $L$ and the probability $P\{L > \nu\}$. If the indemnity is large on “bad” states (when $X$ is large), then the loss $L$ is small on the “bad” states. However, because $E[L]$ is fixed, then there might have large loss on “good” states. On the other hand, if the indemnity is small on “bad” states, then loss is large on “bad” states. Consequently, there exists small loss on “good” states. Because of the risk aversion of the insured, the optimal indemnity is to have small loss on “good” states, and henceforth large loss on “bad” states.

Both Proposition 1.1 and 2.1 are partially consistent with the empirical findings of Froot (2001). Froot (2001) finds that most insurers purchase relatively little reinsurance against catastrophes’ risk. Precisely, the reinsurance coverage as a fraction of the loss exposure is very high above the retention (for the medium losses), and then declines with the size of the loss (See Figure 2 in
Froot (2001) for details). Arrow’s optimal insurance theory implies that this kind of reinsurance contract is not optimal. Froot (2001) provides a number of possible reasons for these departures from theory. Proposition 1.1 and 2.1 present another possible explanation of the excess-of-loss layer feature of the reinsurance coverage. According to Proposition 1.1 and 2.1, this pattern of the reinsurance contract profile is partially preserved as there is no insurance at all for large losses in the presence of a constraint on the risk (through VaR or CTE). If the insurer minimizes the solvency risk or the expected loss, it is tempted to focus on the medium losses and not on the large events.

Clearly, Proposition 1.1 and 2.1 only hold under strong assumptions as stated in Problem 1.1 and 2.1. The model is somewhat too simple to fully explain the design of real insurance contracts regarding large losses. This type of contracts is also subject to moral hazard since companies might hide partly their large losses. An extension of our model to include the interests of debtholders might be possible to overcome these difficulties. In the absence of asymmetric information, debtholders (policyholders and bondholders) of the insurance company would clearly dislike the contract described in Proposition 1.1 and 2.1; they would either refuse to participate or will at least require a risk premium to participate. Therefore policyholders ask for a smaller insurance premium and debtholders require larger interests. The fact that insurers purchase coverage for large losses might thus partially explain by the presence of asymmetric information. Since this paper focuses on the effects of the risk measures, we do not model the asymmetric information between the debtholders and the managers (shareholders). Rather, we wonder whether there is any risk measure leading to other type of optimal reinsurance design.

In the next subsection, we show that a stronger regulatory requirement may provide incentives to purchase insurance against large losses. In fact, we prove that deductible policies become optimal for the insurer when an alternative risk measure is imposed. Thus Arrow’s result on the optimality of deductibles holds for risk neutral agents subject to a risk constraint based on the expected square of the loss for instance.

### 2.2 Emphasize the Right Tail Distribution

The risk measure we consider in this subsection is based on the expected square of the excessive loss. This risk measure is related to the variance tail measure and thus useful when the variability of the loss is high. Precisely, we study the following problem for insurer.

**Problem 2.2** Find the optimal indemnity $I(X)$ that solves:

$$
\min_{I(X)} \left\{ \mathbb{E} \left[ (W_0 - W - \nu)^2 \mathbb{1}_{W_0 - W > \nu} \right] \right\} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ (1 + \rho)\mathbb{E} \left[ I(X) \right] \leq \Delta \end{cases}
$$

\[\text{For more details on this risk measure we refer to Furman and Landsman (2006).}\]
The objective of Problem 2.2 is to minimize the square of excess loss. Comparing with CTE, this risk measure $E\left[(W_0 - W - \nu)^2 1_{W_0 - W > \nu}\right]$ pays more attention on the loss amount over the loss states $\{W_0 - W > \nu\}$. Then, it is termed as “expected square of excessive loss measure”. 

**Proposition 2.2** Assuming $X$ has a continuous cumulative distribution function strictly increasing on $[0, +\infty)$. Assume that $P \in S$. Let $d_P$ be the deductible level whose corresponding deductible contract has premium $P$. Then the solution of Problem 2.2 is a deductible indemnity $(X - d_P^*)^+$, where $P^*$ solves the following minimization problem:

$$\min_{0 \leq P \leq \Delta} E \left[(W_0 - W_P - \nu)^2 1_{W_0 - W_P > \nu}\right]$$

(9)

where $W_P$ is the corresponding wealth of purchasing the deductible $(X - d_P)^+$.

Proof of Proposition 2.2 is presented in Appendix A. In contrast with Propositions 1.1 and 2.1, Proposition 2.2 states that deductibles are optimal when a constraint on the expected square of excessive loss is imposed. Let us explain why this is the case briefly. In term of the loss $L = W_0 - W$, this problem becomes:

$$\min E[L^2 1_{L > \nu}]$$

subject to $E[L]$ is fixed and $P \leq L \leq P + X$. In contrast with Problem 2.1, the objective function in Problem 2.2 involves the square of $L$ which dominates the constraint $E[L]$. Then, intuitively, the optimal indemnity should minimize the loss $W_0 - W$ over the bad states as small as possible. Hence the optimal indemnity is deductible. Proposition 2.2 verifies this intuition.

By Proposition 1.1, 2.1 and 2.2 all together, the choice of the risk measure impacts the optimal reinsurance contract. These results display the market effects of the regulatory constraints by investigating the profile of the optimal reinsurance contract. Moreover, we observe that stronger regulatory control on the risk exposure leads to a better protection of the policyholders and better stability on the market. This is due to the fact that insurers hedge the risk of insolvencies arising from exposure to large losses. In this paper, the expected square of the loss measure is used as an example to show that different optimal reinsurance contracts might follow from different regulatory requirements. We illustrate a methodology that can help authorities to understand consequences of their decisions as well as to choose the appropriate risk measure to regulate the insurance market. More comparisons between these risk measures will be presented in Section 3.

### 3 Optimal Indemnity with Financing Imperfections

We have shown that risk neutral insurers behave risk averse because of the enforcement of risk measure constraints. The profile of the optimal reinsurance contract depends on how the risk control

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13 This measure is not a coherent risk measure in the sense of Artzner et al. (1999) though.

14
policy is requested and implemented. Other factors, mentioned earlier in the literature, contribute to the risk averse attitude of risk neutral insurance companies. In this section, we compare our approach with two alternatives to generate the reinsurance demand by Froot, Scharfstein and Stein (1993) and Caillaud, Dionne and Jullien (2000). Both show that external financing generates an insurance demand by risk neutral firms. This amounts to comparing effects of enforcement regulation constraints and voluntary risk management.

Froot, Scharfstein and Stein (1993) consider a value-maximizing insured facing financing imperfections increasing the cost of the raising of external funds. The imperfections include cost of financial distress, taxes, managerial motives or other capital market imperfections. These imperfections alter the shape of the value function $U(W)$ of the firm where $W$ denotes the internal capital. Under a fairly general condition of the loss $X$, Froot, Scharfstein and Stein (1993) prove that the partial differentiates $U_{WW} < 0, U_W \geq 1$ by using a costly-state verification model of external financing (See Townsend (1979)). Purchasing a reinsurance contract $I(X)$ by paying the premium $P$, the final wealth writes as: $W = W_0 - X + I(X) - P$. The insurer makes a reinsurance decision by maximizing the expected value of the firm $E[U(W)]$. Therefore, the risk neutral firm behaves like a risk averse individual with concave utility function $U(\cdot)$. It has been showed by Arrow (1963) that deductible indemnity is optimal for a risk averse individual. Hence, the optimal reinsurance contract in Froot, Scharfstein and Stein (1993)’s framework is a deductible (no insurance for small losses, full insurance above the deductible level).

Caillaud, Dionne and Jullien (2000) examine the problem from a different angle by rationalizing the use of insurance covenants in financial contracts, say corporate debts. In Caillaud, Dionne and Jullien (2000), external funding for a risky project can be affected by an accident during its realization. Since accident losses and final returns are private information and can be costly evaluated by outside investors, the optimal financial contract must be a bundle of a standard debt contract and an insurance contract which involves full coverage above a straight deductible. Hence, small loss is not insured because of the auditing costs and the bankruptcy costs.

Figure 3 displays the optimal insurance contract based on either regulatory constraints or costly external funding. On Figure 3, the two indemnities have the same actuarial value, thus the same premium. The truncated deductible indemnity is optimal for VaR or CTE constraints, while the deductible indemnity is optimal for either the square of the expected loss risk measure or voluntary risk management.

We compare the risk measure constraint and the voluntary risk management policy. In all cases, small losses stay uninsured (that reduces costs and moral hazard). But there are significant differences on the medium loss and large loss. A Value-at-Risk constraint or a CTE constraint is not enough to induce insurers to protect themselves against large loss amounts. The probability of having a large loss is controlled but the amount of the loss is not. This is the opposite to the deductible indemnity. In the latter case, companies protect large losses up to a fixed amount.

\footnote{Firms makes financing decision and insurance decision to increase firm value. Hence the risk management policy is voluntary comparing with the compulsory risk management requirement imposed by regulator.}
Indemnity $I(x)$ w.r.t. the loss $X$, controlling also their default risk. Strong risk control such as the square of the expected loss risk measure provide incentive to insured large loss, hence the optimal indemnity under this risk measure is identical with the one under voluntary risk management policy.

Regulatory requirement and firm’s risk management policy lead to different protection (or hedging) strategy. VaR and CTE risk management policies provide a better protection on medium losses. If the company only implement the enforced constraint without doing a risk averse risk management, it will benefit on average until a large loss occurs. On the other hand, the firm’s risk management policy focuses on the large loss but this “over hedge” strategy on the large loss reduces the gain on the medium loss level. Thus the enforcement of VaR and CTE regulations will be efficient only in the presence of firm’s risk management program (such as the one suggested by Froot et al. (1993) or Caillaud et al. (2000)).

We now move to the reinsurance contracts in the marketplace.

4 Reinsurance and Capital Market

In this section, we first compare our results to traditional reinsurance policies, then interpret reinsurance arrangements as a derivatives portfolio written on a loss index.

We find that the optimal reinsurance contract is not available in the reinsurance market due to moral hazard issues. However, the optimal strategy under VaR can possibly be implemented in the capital market, as soon as a reference index strongly correlated to the insurer’s loss is traded.

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15 The optimal insurance contract design combining regulatory constraints and costly external funding cost (that is for a risk averse firm) is beyond the scope of this paper.
4.1 Comparing Existing Reinsurance Contracts and our Results

Froot (2001) underlines that most reinsurance arrangements are “excess-of-loss layer” with a retention level (the deductible level that losses must exceed before coverage is triggered), a limit (the maximum amount reimbursed by the reinsurer) and an exceeding probability (probability losses are above the limit). The contract writes as:

\[ I_1(X) = (X - d) \mathbb{1}_{X \in [d,l]} + (l - d) \mathbb{1}_{X > l}. \]  

(10)

where \( d \) is the deductible level, \( l \) stands for the upper limit of the coverage, thus \( l - d \) is the maximum indemnity. This contract is a popular contract involving a stop loss rule (that is a deductible) with an upper limit on coverage \[.]^{16}

Let us compare it with our previous results. We already pointed out that the optimal design might induce moral hazard, in particular insurers have incentive to partly hide their losses. A contract such as the one given by expression \[.]\(10\) does not have this drawback since the indemnity is non-decreasing with the loss amount. Figures 4 illustrates a comparison of these two designs with arbitrary parameters.

![Figure 4: Indemnity \( I(X) \) w.r.t. \( X \)](image)

We compare the optimal contract and the deductible with an upper limit. We assume the premium \( P \) is the same in both contracts. Thus, if one contract reimburses more than another one for one range of loss amounts, then it must be the contrary for another range of losses amounts. On Figure 4 the plain line corresponds to the optimal contract under a VaR constraint and the dash line is the capped contract. They have the same premium \( P \), so the coverage provided by the optimal contract is better for medium losses but worse for extreme losses.

\[\text{Policies with upper limit on coverage could be derived from minimizing some risk measures under a mean variance premium principle. They also have been extensively analyzed by Cummins and Mahul (2004).}\]
Assuming that the company cannot manipulate the actual amount of loss, the truncated deductible is then a possible design and seems to be a better design. For a similar amount of money spent in reinsurance, the company obtains a better coverage for medium losses which are the most possible losses but a lower for extreme losses. Moreover, the coverage provided by the upper-limit contract is fixed in the case when an extreme loss occurs (equal to the level of the upper-limit). Thus, such a contract does not avoid bankruptcy.

Moral hazard can be also reduced by a coinsurance treaty, where the coverage is partly provided by the reinsurer (for instance $\theta = 95\%$) (see Cummins, Lalonde and Phillips (2004))

$$I_{\theta}(X) = \theta(X - d)1_{X\in[d,l]} + \theta(l - d)1_{X>l}, \quad \theta < 1.$$ 

The recent work of Froot (2001) gives an overview of the market for catastrophe risk. He notices that “most insurers purchase relatively little cat reinsurance against large events and that premiums are high relative to expected losses”. He explains that both reinsurance and CAT bonds generally trade at significant margins above the expected loss and that insurers tend to retain rather than share their large event risks. This is in concordance with our study. Truncated deductible are indeed optimal when risk constraints are linked to a VaR or a CTE constraint. Then insurance companies choose to let uninsured the worst possible states.

### 4.2 Alternative on Capital Market

Recently, it has become widely appreciated that a single natural hazard could result in damages of several billions. For instance the total insured US catastrophic losses for 2005 are estimated to be more than $50 billion, where the three major hurricanes Katrina, Rita and Wilma make up 90% of the total loss of the year\(17\). Even if the insurance industry’s equity capital would be enough to absorb lots of catastrophic events, Cummins, Doherty and Lo (2002) explain that many insurers can become insolvent depending on the distribution of damage and their portfolio of policies. They note that in absence of costs, the Pareto optimal way to share risks is to mutualize all risks between all insurers.

The traditional instrument to spread risks between insurers is reinsurance. By reinsuring a layer of one line of business or of a specific risk, insurers buy and sell options on the loss index. Assuming risks can be measured in terms of an index (for instance, temperature, a specified event or wind speed), then a reinsurance arrangement can be interpreted as a portfolio of derivatives written on this underlying index.

More precisely, Froot (2001) and Cummins, Lalonde and Phillips (2004) compare reinsurance layers with call spreads. They explain how insurers hedge their risks by forming a portfolio consisting in its losses (assume to be traded as a loss index $X$) and a position in call option spreads on the

\[17\]Source: The annual report Guy Carpenter, “US Reinsurance Renewals at January 1, 2006”
where $\theta$ corresponds to a coinsurance treaty. The indemnity is:

$$I_\theta(X) = \begin{cases} 
0 & \text{If } X < d \\
\theta(X - d) & \text{If } X \in [d, l] \\
\theta(l - d) & \text{If } X > l 
\end{cases}$$

In presence of regulators’ minimum capital requirement and VaR or Conditional Tail expectation maximum risk exposure, we show that the optimal reinsurance arrangement is:

$$I(X) = (X - d)\mathbb{1}_{X \in [d, q]}$$

where $d$ is the deductible level and $q$ the upper limit. Then,

$$I(X) = (X - d)^+ - (X - q)^+ - (q - d)\mathbb{1}_{X > q}$$

which corresponds to a portfolio of derivatives, a long position on a call and a short position on a put and on a barrier bond (activated when the underlying $X$ is above $q$). Equivalently, the company possesses:

$$X - I(X) = X - [(X - d)^+ - (X - q)^+ - (q - d)\mathbb{1}_{X > q}] .$$

Comparing this portfolio with the call spread (11), we show it is optimal for companies to sell the bond corresponding to the right tail risk. In some sense, they optimally give up the right tail, because there is no reason to pay for (useless) reinsurance after being ruined (when the upper-limit on coverage is not enough to stay solvent).

Using a simulation model, Cummins, Lalonde and Phillips (2004) study the efficiency of hedging with reinsurance or index-linked securities. They explain that reinsurance contracts are sold over the expected loss and that it is less efficient than hedging using contracts actuarially fairly priced. Insurance-linked securities are mostly competitive with reinsurance in terms of price and hedge efficiency since they can be traded at significantly lower margins. Duplicate reinsurance arrangements on the market is a worth alternative risk transfer.

**Conclusions**

New risk management programs are currently being implemented to the insurance market, for example the Solvency II project. We focus on risk-neutral insurance companies subject to several tail risk measures imposed by regulators. We derive the design of the optimal reinsurance...
contract to maximize the expected profit when the regulatory constraints are satisfied. We show that insurance companies have no incentives to protect themselves against extreme losses when regulatory requirements are based on Value-at-Risk or Conditional Tail Expectation. These results may partially confirm observed behaviors of insurance companies (Froot (2001)). Furthermore, we show that an alternative risk measures would lead insurance companies to fully hedge the right tail of the loss distribution.

The model in this paper is quite simple. There are no transaction cost for issuing and purchasing reinsurance contracts, no background risk, and a single loss during the period of insurance protection. Moreover both issuer and issued are risk neutral, both parties have symmetric (and perfect) information about the distribution of the loss. Even with the previously mentioned model limitations, the results of this paper could still be used as “prototypes” by insurance companies to design optimal risk management strategies, as well as by regulators to impose appropriate risk measures. Because of the similarities between the reinsurance market and the capital market, our results also present alternative risk transfers mechanisms in the capital market.
A Proofs

Recall that final wealth $W$ is given by $W = W_0 - P - X + I(X)$. Then, the event $\{W \geq W_0 - \nu\}$ is the same as $\{I(X) \geq P + X - \nu\}$ in terms of the coverage $I(X)$.

A.1 Proposition 1.1

The solution is derived as follows. The first step is to derive the optimal reinsurance coverage when the premium is fixed (Problem A.1 below). In the second step Problem 1.1 is reduced to a sequence of Problem A.1. As we have discussed in the main body of the text, the rationale of this approach follows from the property that $L(P)$ is not monotone, consequently, the premium constraint is not necessary binding.

Problem A.1 Find the optimal reinsurance indemnity such that

$$\min_{I(X)} P\{W < W_0 - \nu\} \quad \text{s.t.} \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X) + C(I(X))] = P \end{cases}$$

Equivalently, Problem A.1 is reformulated as follows.

$$\max_{I(X)} P\{I(X) \geq P + X - \nu\} \quad \text{s.t.} \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X) + C(I(X))] = P \end{cases}$$

Lemma A.1 If $Y^*$ satisfies the three following properties:

(i) $0 \leq Y^* \leq X$,

(ii) $E[Y^* + C(Y^*)] = P$,

(iii) There exists a positive $\lambda > 0$ such that for each $\omega \in \Omega$, $Y^*(\omega)$ is a solution of the following optimization problem:

$$\max_{Y \in [0, X(\omega)]} \left\{ 1_{P + X(\omega) - \nu \leq Y} - \lambda(Y + C(Y)) \right\}$$

then $Y^*$ solves the optimization problem A.1.

Proof. Given a coverage $I$ which satisfies the constraints of the optimization problem A.1. Therefore, using (iii), we have,

$$\forall \omega \in \Omega, \quad 1_{P + X(\omega) - \nu \leq Y^*} - \lambda(Y^* + C(Y^*)) \geq 1_{P + X(\omega) - \nu \leq I(\omega)} - \lambda(I(\omega) + C(I(\omega)))$$

Thus,

$$1_{P + X(\omega) - \nu \leq Y^*(\omega)} - 1_{P + X(\omega) - \nu \leq I(\omega)} \geq \lambda(Y^*(\omega) + C(Y^*(\omega)) - I(\omega) - C(I(\omega))) .$$

We now take the expectation of this inequality. Therefore by condition (ii) one obtains,

$$P\{P + X - \nu \leq Y^*\} - P\{P + X - \nu \leq I\} \geq \lambda(P - E[I + C(I)])$$

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Therefore, applying the constraints of the variable $I$, $\mathbb{E}[I(X) + C(I(X))] = P$

$$P\{P + X - Y^* \geq \nu\} - P\{P + X - I \geq \nu\} \geq 0$$

The proof of this lemma is completed. \hfill \Box

**Lemma A.2** When $P \leq \nu$, each member of the following family $\{Y_\lambda\}_{\lambda > 0}$ satisfies the conditions (i) and (iii) of Lemma A.1.

$$Y_\lambda(\omega) = \begin{cases} 
0 & \text{if } X(\omega) < \nu - P \\
X(\omega) + P - \nu & \text{if } \nu - P \leq X(\omega) \leq \nu - P + D\left(\frac{1}{\lambda}\right) \\
0 & \text{if } X(\omega) > \nu - P + D\left(\frac{1}{\lambda}\right) 
\end{cases}$$

where $D$ is the inverse of $y \to y + C(y)$.

Proof. The property (i) is obviously satisfied. Indeed we only study the case when $\nu$ is more than the premium $P$.

First, if $X(\omega) < \nu - P$, then $P + X(\omega) - \nu < 0$, the function to maximize over $[0, X(\omega)]$ is equal to $1 - \lambda(Y + C(Y))$, decreasing over the interval $[0, X(\omega)]$ (since $C'(\cdot) > -1$), the maximum is thus obtained at $Y^*(\omega) = 0$.

Otherwise, $X(\omega) \geq \nu - P$. Since $P \leq \nu$, one has $P + X(\omega) - \nu \leq X(\omega)$. We consider two cases: firstly, if $Y \in [0, P + X(\omega) - \nu)$, then the function to maximize is $-\lambda(Y + C(Y))$. It is decreasing with respect to the variable $Y$. Its maximum is 0, obtained at $Y = 0$. Secondy, if $Y \in [P + X(\omega) - \nu, X(\omega)]$, then the function to maximize is $1 - \lambda(Y + C(Y))$. It is decreasing. Its maximum is obtained at $Y = P + X(\omega) - \nu$ and its value is $1 - \lambda(P + X(\omega) - \nu + C(P + X(\omega) - \nu))$.

We compare the value $1 - \lambda(P + X(\omega) - \nu + C(P + X(\omega) - \nu))$ and 0 to decide whether the maximum is attained at $Y = X(\omega) + P - \nu$ or $Y = 0$.

$$1 - \lambda(P + X(\omega) - \nu + C(P + X(\omega) - \nu)) \geq 0 \quad \iff \quad P + X(\omega) - \nu + C(P + X(\omega) - \nu) \leq \frac{1}{\lambda}.$$ 

Let $D = (Y + C(Y))^{-1}$ that exists since $Y + C(Y)$ is increasing, then:

$$X(\omega) \leq \nu - P + D\left(\frac{1}{\lambda}\right).$$

Lemma A.2 is proved. \hfill \Box

**Proof of Proposition 1.1** Thanks to lemmas A.1 and A.2, it suffices to prove that there exists $\lambda > 0$ such that $Y_\lambda$ defined in lemma A.2 satisfies the condition (ii) of lemma A.1. We then compute its associated cost function.

$$\mathcal{E}_\lambda := \mathbb{E}\left[(X + P - \nu)1_{X \in \left[\nu - P, \nu - P + D\left(\frac{1}{\lambda}\right)\right]} + C\left((X + P - \nu)1_{X \in \left[\nu - P, \nu - P + D\left(\frac{1}{\lambda}\right)\right]}\right)\right].$$

It is obvious then:

$$\lim_{\lambda \to 0^+} \mathcal{E}_\lambda = \mathbb{E}\left[(X - \nu + P)^+ + C\left((X - \nu + P)^+\right)\right], \quad \lim_{\lambda \to +\infty} \mathcal{E}_\lambda = 0.$$
By Lebesgue dominance theorem we can easily prove the convergence property of \( \mathcal{E}_\lambda \) with respect to the parameter \( \lambda \). Then the existence of a solution \( \lambda^*_P \in \mathbb{R}_+^\ast \) such that \( \mathcal{E}_\lambda = P \) comes from the assumption on the continuous distribution of \( X \) and thus the continuity of \( \mathcal{E}_\lambda \). Thus we have proved the first part of this Proposition. The second part follows easily from the first part. \( \square \)

A.2 Proposition 2.1

The solution of Problem 2.1 consists of two steps. We first solve Problem 2.1 by fixing the premium, reducing it to Problem A.2 below.

Problem A.2 Solve the indemnity such that

\[
\min_{I(X)} \left\{ \mathbb{E} \left[ (W_0 - W) \mathbb{I}_{W_0 - W > \nu} \right] \right\} \quad \text{s.t.} \quad \begin{cases} 0 \leq I(X) \leq X \\ (1 + \rho) \mathbb{E} [I(X)] = \Delta \end{cases}
\]

Equivalently, Problem 2.1 is to minimize \( \mathbb{E} \left[ (P + X - I) \mathbb{1}_{P + X - I > \nu} \right] \) subject to the same constraints. Because of the linear premium principle, for the sake of simplicity we ignore the loading factor in the remainder proofs of Appendix A.

Lemma A.3 If \( Y^\ast \) satisfies the three following properties:

(i) \( 0 \leq Y^\ast \leq X \),

(ii) \( \mathbb{E} [Y^\ast] = \Delta \),

(iii) There exists a positive \( \lambda > 1 \) such that for each \( \omega \in \Omega \), \( Y^\ast(\omega) \) is a solution of the following optimization problem:

\[
\min_{Y \in [0, X(\omega)]} \left\{ (P + X(\omega) - Y) \mathbb{1}_{Y < P + X(\omega) - \nu} + \lambda Y \right\}
\]

then \( Y^\ast \) solves Problem A.2.

Proof. The proof of Lemma A.3 is similar to the proof of Lemma A.1. \( \square \)

Proof of Proposition 2.1 We use Lemma A.3 and show that for \( \lambda > 1 \),

\[
Y_\lambda(\omega) = \begin{cases} 0 & \text{if } X(\omega) < \nu - P \\
X(\omega) + P - \nu & \text{if } \nu - P \leq X(\omega) \leq \nu - P + \frac{\nu}{\lambda - 1} \\
0 & \text{if } X(\omega) > \nu - P + \frac{\nu}{\lambda - 1} \end{cases}
\]

satisfies conditions (i) and (iii) of Lemma A.3. If \( X(\omega) + P - \nu < 0 \) then \( Y^\ast = 0 \). Otherwise \( 0 \leq P + X(\omega) - \nu < X \). Similar to the proof of Proposition 1.1 we can prove that \( Y = P + X(\omega) - \nu \) is the maximum one if \( \nu - P \leq X(\omega) \leq \nu - P + \frac{\nu}{\lambda - 1} \), else the maximum one is \( Y = 0 \) if \( X(\omega) > \nu - P + \frac{\nu}{\lambda - 1} \).

Let

\[
\mathcal{E}_\lambda := \mathbb{E} \left[ (X + P - \nu) \mathbb{1}_{X \in (\nu - P, \nu - P + \frac{\nu}{\lambda - 1})} \right].
\]
It is obvious then:
\[
\lim_{\lambda \to 1^+} E_\lambda = E \left[ (X - \nu + P)^+ \right], \quad \lim_{\lambda \to +\infty} E_\lambda = 0.
\]
The existence of a solution \( \lambda^* > 1 \) such that \( E_\lambda = \Delta \) follows from the assumption on the continuous distribution of \( X \) and thus the continuity of \( E_\lambda \). Therefore, we have proved the first part of Proposition 2.1. The second part follows easily from the first part. \( \square \)

A.3 Proposition 2.2

Problem A.3 Find the optimal indemnity that solves:
\[
\min_{I(X)} \left\{ E \left[ (W_0 - W - \nu)^2 1_{W_0 - W > \nu} \right] \right\} \quad s.t. \quad \begin{align*}
0 & \leq I(X) \leq X \\
E[I(X)] & = \Delta
\end{align*}
\]

Lemma A.4 If \( Y^* \) satisfies the three following properties:
(i) \( 0 \leq Y^* \leq X \),
(ii) \( E[Y^*] = \Delta \),
(iii) There exists a positive \( \lambda > 0 \) such that for each \( \omega \in \Omega \), \( Y^* \) is a solution of the following optimization problem:
\[
\min_{Y \in [0, X(\omega)]} \left\{ (P + X(\omega) - Y - \nu)^2 1_{Y < P + X(\omega) - \nu} + \lambda Y \right\}
\]
than \( Y^* \) solves Problem A.3.

Proof. Let \( Y^* \) be a random variable satisfying the three above conditions of the lemma. On the other hand, given another available payoff \( I \) which satisfies the constraints of the above optimization problem. Therefore, using (iii), we have, \( \forall \omega \in \Omega \),
\[
(P + X(\omega) - Y^*(\omega) - \nu)^2 1_{Y^*(\omega) < P + X(\omega) - \nu} + \lambda Y^*(\omega) \leq
(P + X(\omega) - I(\omega) - \nu)^2 1_{I(\omega) < P + X(\omega) - \nu} + \lambda I(\omega)
\]
Thus,
\[
(P + X(\omega) - Y^*(\omega) - \nu)^2 1_{Y^*(\omega) < P + X(\omega) - \nu} - (P + X(\omega) - I(\omega) - \nu)^2 1_{I(\omega) < P + X(\omega) - \nu} \leq \lambda (I(\omega) - Y^*(\omega))
\]
We now take the expectation of the above inequality, therefore by condition (ii) one obtains,
\[
E \left[ (P + X - Y^* - \nu)^2 1_{Y^* < P + X - \nu} \right] - E \left[ (P + X - I - \nu)^2 1_{I < P + X - \nu} \right] \leq \lambda (E[I] - \Delta)
\]
Therefore, applying the constraints of the variable \( I \), \( E[I(X)] = \Delta \),
\[
E \left[ (P + X - Y^* - \nu)^2 1_{Y^* < P + X - \nu} \right] \leq E \left[ (P + X - I - \nu)^2 1_{I < P + X - \nu} \right]
\]
The proof of this lemma is completed. \( \square \)
Lemma A.5 When \( P \leq \nu \), each member of the following family \( \{Y_\lambda\}_{\lambda>0} \) satisfies the conditions (i) and (iii) of Lemma A.4:

\[
Y_\lambda(\omega) = \begin{cases} 
0 & \text{if } X(\omega) < \nu - P + \frac{\lambda}{2} \\
X(\omega) + P - \nu - \frac{\lambda}{2} & \text{if } \nu - P + \frac{\lambda}{2} \leq X(\omega)
\end{cases}
\]

Proof. The property (i) is obviously satisfied. We now prove the property (iii).

First, if \( X(\omega) < \nu - P \), then \( P + X(\omega) - \nu < 0 \), the function to minimize over \([0, X(\omega)]\) is equal to \(\lambda Y\), increasing over the interval \([0, X(\omega)]\), the minimum is thus obtained at \(Y^*(\omega) = 0\).

Otherwise, \( X(\omega) \geq \nu - P \). Since \( P \leq \nu \), one has \( P + X(\omega) - \nu \leq X(\omega) \). Thus we have to solve the optimization problem under the assumption \( 0 \leq P + X(\omega) - \nu \leq X(\omega) \). There are two cases: firstly, if \( Y \in [0, P + X(\omega) - \nu) \), then the function to minimize is

\[
\phi_1(Y) = (P + X(\omega) - \nu - Y)^2 + \lambda Y.
\]

Its minimum is \( \max \{0, X(\omega) + P - \nu - \frac{\lambda}{2}\} \). Secondly, if \( Y \in [P + X(\omega) - \nu, X(\omega)] \), then the function to minimize is

\[
\phi_2(Y) = \lambda Y.
\]

Its minimum is obtained at \( Y = P + X(\omega) - \nu \) and its value is \( \lambda(P + X(\omega) - \nu) \). We then compare this value with the previous minimum:

- When \( 0 < X(\omega) + P - \nu - \frac{\lambda}{2} \), \( \Phi_1(X(\omega) + P - \nu - \frac{\lambda}{2}) = \Phi_2(P + X(\omega) - \nu) - \frac{\lambda^2}{4} < \Phi_2(P + X(\omega) - \nu) \).
- When \( 0 > X(\omega) + P - \nu - \frac{\lambda}{2} \), \( \Phi_1(0) = (P + X(\omega) - \nu)^2 \). Since \( \frac{\lambda}{2} > X(\omega) + P - \nu \), \( \Phi_1(0) < \frac{\Phi_2(P + X(\omega) - \nu)}{2} < \Phi_2(P + X(\omega) - \nu) \).

Obviously, the minimum is thus obtained when \( Y = \max \{0, X(\omega) + P - \nu - \frac{\lambda}{2}\} \). Lemma A.5 is proved.

Proof of Proposition 2.2.

Thanks to both Lemmas A.4 and A.5, one only has to prove that there exists \( \lambda > 0 \) such that \( Y_\lambda \) defined in Lemma A.5 satisfies the condition (ii) of Lemma A.4. We then compute its expectation.

\[
\mathcal{E}_\lambda := E \left[ \left(X + P - \nu - \frac{\lambda}{2}\right) 1_{X \in [\nu - P + \frac{\lambda}{2}, +\infty)} \right].
\]

We see that

\[
\lim_{\lambda \to 0^+} \mathcal{E}_\lambda = E [(X - \nu + P)^+] \quad \text{and} \quad \lim_{\lambda \to +\infty} \mathcal{E}_\lambda = 0.
\]

The existence of a solution \( \lambda^* \in \mathbb{R}_+^* \) such that \( \mathcal{E}_\lambda = \Delta \) comes from the assumption on the continuous distribution of \( X \) and thus the continuity of \( \mathcal{E}_\lambda \). Thus we have proved the first part of this Proposition. The second part follows easily from the first part.
References


