Optimal Reinsurance Arrangements
Under Tail Risk Measures

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Keywords: Optimal Reinsurance, Risk Measures, Alternative Risk Transfer

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Abstract

Regulatory authorities demand insurance companies to control their risk exposure by imposing stringent risk management policies. This article investigates the optimal risk management strategy of an insurance company subject to regulatory constraints. We provide optimal reinsurance contracts under different tail risk measures and analyze the impact of regulators’ requirements on risk-sharing in the reinsurance market. Our results underpin adverse incentives for the insurer when compulsory risk management requirements are imposed. Moreover, our findings confirm recent empirical studies of Froot (2001), in which most insurer purchase partial reinsurance against large loss. We propose an alternative risk measure that might be more appropriate to request insurance companies to deal with the risk of possible large losses. Finally, we compare the obtained optimal designs to existing contracts and alternative risk transfer mechanisms on the capital market.

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Introduction

Risk averse individuals purchase insurance policies to increase the expected utility of their final wealth. Arrow (1963, 1971), Borch (1971), Raviv (1979) give the fundamental principles of optimal insurance design in an expected utility framework. This expected utility model has been extensively extended by many authors such as Gollier (1996), Doherty and Schlesinger (1983, 1985), and has been successfully used to explain the insurance demand for risk averse individuals. However, the expected utility framework could not justify the insurance demand for risk neutral firms (see Eeckhoudt, Gollier and Schlesinger (2005)). Indeed, at first sight, risk neutral firms have no desire to buy costly insurance since expected utility of their final wealth would be decreased by purchasing an insurance contract and firms could eliminate insurable risk through diversification. In this paper we examine the prevalence of reinsurance demand and derive optimal risk management strategies for risk neutral insurance companies.

There have been many previous studies to explain why risk neutral corporations purchase insurance based on modern finance theory. Mayers and Smith (1982) might be the first ones to recognize that insurance purchases are part of firm’s financing decision and to explain how insurance demand can reduce regulatory constraints on firms. The findings in Mayers and Smith (1982) have been empirically supported or extended in the literature. Yamori (1999) empirically observe that Japanese corporations can have a low default probability and a high demand for insurance. Davidson, Cross and Thornton (1992) show that the corporate purchase of insurance lies in the bondholder’s priority rule. Hoyt and Khang (2000) argue that corporate insurance purchases are driven by agency conflicts, tax incentives, bankruptcy costs and regulatory constraints. Hau (2006) show that liquidity is important for property insurance demand. In a series of interesting papers, Froot, Sharfstein and Stein (1993), Froot and Stein (1998) and Froot (2001) explain the firm would behave risk averse because of the risk management policies that are used to address the presence of costs.

In this paper, we undertake a different view on the reinsurance demand by focusing on the impact of compulsory risk management policies. Regulators introduce several risk management methodologies (such as Solvency II) to protect policyholders and monitor closely how insurance companies implement those risk management programs. For this purpose, regulators often impose some risk measures and regulated insurance companies are enforced to satisfy these risk measure constraints along their business lines. Thus, in our setting, reinsurance demand is directly linked to the presence of regulators. It turns out that in the presence of regulations, risk neutral insurance companies partly behave as risk averse agents. This fact has been recognized by Caillaud, Dionne and Jullien (2000) in another context, in which they study the optimal financial contract to invest in a risky project.

We derive the optimal reinsurance coverage subject to risk measure constraints imposed by regulators. Our contribution in this paper is to analyze the impact of the choice of the risk measure constraint on the optimal risk-sharing in the reinsurance market. Con-

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1Other related references include Chen and PonArul (1989), Seog (2006) and Plantin (2006).
sequently, reinsurance demand for risk neutral insurance company is derived because of the presence of risk measure constraints.

More precisely, there are three major findings in this paper. First, we present optimal risk management strategies for insurance companies by incorporating the compulsory risk management constraints. Second, we discuss the impact of regulators’ choice on the optimal coverage of the reinsurance contracts and assess regulators’ influence on the risk management decisions taken by companies. We show how adverse incentives can come from tail risk measures such as Value-at-Risk (VaR) or Conditional Tail Expectation (CTE). We see that there is no different optimal risk management strategies when CTE must be satisfied enforcedly. Our findings confirm recent empirical studies (for instance Froot (2001)) which show that insurers do not often purchase coverage for high layers of risk. Third, we show how a stronger control of regulators can lead to optimality of deductibles. We finally compare our results with actual contracts observed in both the reinsurance market and the capital market.

The organization of this paper is as follows. In the next section, a theoretical framework is introduced, and we derive the optimal reinsurance contract under VaR risk measure. Further technical issues are discussed closely in the subsequent section heading with “Dual Problem”. In the “Optimal Reinsurance Arrangements under CTE and other Risk Measures” section we solve the optimal reinsurance design problem under other risk measures. We show that the optimal coverage under CTE measure is similar to the optimal coverage under VaR measure. However, under an alternative risk measure, a deductible contract is optimal. Then we compare the optimal reinsurance design with others in previous literature when the firm behaves risk averse in other frameworks. Our comparison is presented in “Optimal Indemnity with Financing Imperfections”. In the “Reinsurance Market, Capital Market” section we compare the optimal reinsurance contracts under risk measures to contracts frequently sold by reinsures in the marketplace. The “Conclusions” section concludes, and the “Appendix” section provides the proofs.

1 Optimal Reinsurance Design under VaR Measure

Let us consider a one-period model. At the initial time, the insurer receives premia from its customers, and in exchange to these premia, it has to provide its customers a coverage at the end of the period. The aggregate amount of indemnities paid in the future is denoted by $X$. Its initial wealth $W_0$ is composed by the collected premia and its own capital. Its final wealth, at the end of the period, is $\hat{W} = W_0 - X$ if no reinsurance is purchased. The insurer is assumed to be risk-neutral.

Since some risks cannot be diversified (e.g. longevity risk, catastrophic risk), we suppose the insurer faces a risk of large loss. It is requested by regulators to meet risk

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2The idea of the Conditional Tail Expectation (CTE) is to capture not only the probability to incur a high loss but also its magnitude. From a theoretical perspective CTE is better than VaR (see Artzner et al. (1999), Inui and Kijima (2005)), and it has already been implemented to regulate some insurance products (with financial guarantees) in Canada.
management requirements to protect interests of both policyholders and shareholders. As an example, assume that \( v \) is a VaR limit to confidence level \( \alpha \) imposed by regulators, then the VaR constraint for the insurance company is

\[
\Pr \left\{ W_0 - \hat{W} > v \right\} \leq \alpha.
\]  

(1)

This probability \( \Pr \{ W_0 - \hat{W} > v \} \) measures the *insolvency risk*.

Assume the insurer purchases a reinsurance contract from a reinsurer, paying an initial premium \( \mu \). When \( X \) is observed, an indemnity \( I(X) \) is transferred from the reinsurer to the insurer. Then,

\[
W = W_0 - \mu - X + I(X),
\]

where \( W \) the insurance company’s final wealth after entering the reinsurance market. The indemnity \( I(X) \) is understood as a function of the loss variable \( X \). Following classical insurance literature (see for example Arrow (1971) and Raviv (1979)), the coverage \( I(X) \) is nonnegative and can not exceed the size of the loss.

In the design of the reinsurance agreement, we assume the premium \( \mu \) satisfies that:

\[
\mu = \mathbb{E}[I(X) + C(I(X))]
\]  

(2)

where the cost function \( C(.) \) is nonnegative and satisfies \( C' > -1 \). Note that this assumption is fairly general and include many premium principles as special cases. For instance, Arrow (1963) shows the deductible policy is optimal when the premium depends on the expected payoff of the policy only. Raviv (1979) extends Arrow’s analysis to the convex cost structure. Huberman, Mayers and Smith (1983) introduce concave cost structure and found deductible might be not optimal.

The final loss \( L \) of the insurance company is \( L = W_0 - W = \mu + X - I(X) \) that is the sum of the premium \( \mu \) and the retention of the loss. The VaR constraint in this case is formulated as \( \Pr \{ L > v \} \leq \alpha \). It is equivalent to \( \text{VaR}_L(\alpha) \leq v \). A simple reflection implies that the premium must be smaller than the VaR limit \( v \). Otherwise, say \( \mu > v \), then the loss \( L = \mu + X - I(X) > v \) because \( I(X) \) is nonnegative. Hence, the VaR condition does not meet at all if the insurance company pays a premium \( \mu \) which is greater than the VaR limit \( v \).

The objective in what follows is to solve the following Problem 1.1.

**Problem 1.1** Find a reinsurance contract that minimizes insolvency risk:

\[
\min_{I(X)} \Pr \{ W < W_0 - v \} \quad \text{s.t.} \quad \left\{ \begin{array}{l}
0 \leq I(X) \leq X \\
\mathbb{E}[I(X) + C(I(X))] \leq \Delta
\end{array} \right.
\]

---

\(^3\)In fact, rating agencies currently not only focus on the default risk and the volatility of the insurance company, but also incorporate the risk management of the regulated insurance company. Therefore, the presence of rating agencies also introduces risk aversion to the insurer.

\(^4\)VaR is defined by \( \text{VaR}_L(\alpha) = \inf \{ x, \Pr \{ L > x \} \leq \alpha \} \).
Problem 1.1 solves the minimum insolvency risk, or alternatively, the maximum non-insolvent (or survive) probability that the loss $L$ is less than the VaR limit, by paying at most a premium $\Delta$ to purchase a reinsurance contract. The probability $\Pr\{W < W_0 - v\}$ can be written as an expected utility $E[u(W)]$ with a utility function $u(z) = 1_{z < W_0 - v}$. This utility function, however, is not concave. Therefore, standard Arrow-Raviv first order conditions are not sufficient to characterize the optimum. Note that this remark also applies to subsequent problems investigated in the paper.

There is another nice interpretation of Problem 1.1. Note that $E[W] = W_0 - E[X] - \mu + E[I(X)] = W_0 - E[X] - E[I(X) + C(I(X))] + E[I(X)]$. Therefore,

$$E[I(X) + C(I(X))] \leq \Delta \implies E[W] \geq W_0 - E[X] - \Delta.$$  

Hence, Problem 1.1 partly solves the minimum insolvency probability subject to a minimum guaranteed expected final wealth. Thus, Problem 1.1 represents the trade-off between the minimal guaranteed expected wealth and the insolvency risk measured by the probability that losses exceed the VaR limit.

We add one comment on the concept of optimality to finish this section. In this paper, we focus on the optimal shape of the contract in a Pareto-optimality framework while a premium principle (2) is imposed. The optimal premium level, which is not addressed in this paper, can be solved via a numerical search (See Schlesinger (1981), Meyer and Ormiston (1999)).

### Solution to Problem 1.1

The solution is derived as follows. We first derive the optimal reinsurance coverage when the premium is fixed (Problem 1.2 below). Then Problem 1.1 is reduced to a sequence of Problem 1.2. The rationale of this approach will become clearer after the discussion below and in section 2. Indeed, we show that in some cases the risk measure constraint is not binding.

#### Problem 1.2

Find the optimal reinsurance indemnity such that

$$\min_{I(X)} \Pr\{W < W_0 - v\} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X) + C(I(X))] = \mu \end{cases}$$

Let $S := \{\mu : 0 \leq \mu < E[(X - v + \mu)^+ + C((X - v + \mu)^+)]\}$.

### Proposition 1.1

Assuming $X$ has a continuous cumulative distribution function. Assume that $\mu \in S$. Then there exists a solution:

$$I_\mu(X) = \begin{cases} 0 & \text{if } X < v - \mu \\ X + \mu - v & \text{if } v - \mu \leq X \leq v - \mu + k_\mu \\ 0 & \text{if } X > v - \mu + k_\mu \end{cases}$$

---

5In the literature, there are two separate concepts, one is to determine the optimal shape of the insurance contract and one is to find the optimal level of insurance.

6We allow the case when there is a mass point at 0 meaning $P(X = 0)$ can be positive.
of Problem 1.2, where \( k_\mu > 0 \) satisfies \( E[I_\mu(X) + C(I_\mu(X))] = \mu \). Moreover, define
\[
P(\mu) = \Pr\{X > v - \mu + k_\mu\}, \quad \forall \mu \in S.
\]
Then \( I_{\mu^*}(X), \mu^* \leq \mu \) solves Problem 1.1 where \( \mu^* \) solves the static minimization problem
\[
\min_{0 \leq \mu \leq \Delta} P(\mu).
\]
According to Proposition 1.1, the optimal reinsurance coverage of Problem 1.2 involves a deductible for small loss, and coinsurance (actually no insurance) for large loss. We call it “truncated deductible”. This proposition is intuitive. The objective is to minimize the insolvency probability. The optimal contract must be one in which the indemnity on “bad” states are transformed to the “good” states by keeping the premium (total expectation) fixed. Because reinsurance is costly, it is optimal not to purchase reinsurance for the large loss states.

Assume that \( \alpha \) is the smallest probability in Problem 1.1. Then, by Proposition 1.1, we have:
\[
v - \mu^* + k_{\mu^*} = q
\]
where \( \Pr(X \leq q) = 1 - \alpha \), namely, \( q \) is the \((1 - \alpha)-\)quantile of the distribution of \( X \).
In the following, \( I(X) \) denotes the optimal coverage \( I_{\mu^*}(X) \). The trade-off between the premium \( \mu^* \) and the minimal insolvency probability \( \alpha \) is characterized by the following premium equation:
\[
E \left[(X + \mu^* - v)^+ \mathbb{1}_{X \leq q} + C \left((X + \mu^* - v)^+ \mathbb{1}_{X \leq q}\right)\right] = \mu^*.
\]
Figure 1 displays a typical optimal indemnity \( I(X) \). In Figure 1, the premium \( \mu^* = 2 \), the VaR limit \( v = 5 \), the deductible \( d = v - \mu^* = 3 \) and the \((1 - \alpha)-\)quantile \( q = 7 \). There exists no reinsurance when the loss \( X > q \). The indemnity \( I(X) \) is deductible over the loss state \( \{X \leq q\} \). As a consequence of this proposition, the presence of a Value-at-Risk constraint affects the shape of the optimal contract: worst losses stay optimally uninsured. Hence, if insurer is enforced to implement the Value-at-Risk constraint, the insurer has adverse incentive because uninsured the large loss is optimal. This adverse incentive of insurer follows from special requirement of the VaR risk metrics. We will back to this issue later when we discuss other risk measures and the corresponding optimal reinsurance contracts later.

If the VaR limit \( v \) is set to be the initial wealth \( W_0 \), Problem 1.1 can be interpreted in terms of ruin probability:
\[
\min \Pr\{W < 0\} \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X) + C(I(X))] \leq \Delta \end{cases}
\]
The objective is to minimize the probability of ruin. Proposition 1.1 states that the company gives up to insure itself above a given level. In this special case, this problem is
considered by Gajek and Zagrodny (2004b). Even though they discover the same optimal shape, their construction of the optimal coverage is not explicit.

Since the set $\mathcal{S}$ plays a key role in Proposition 1.1 and subsequent discussions in Proposition 3.1 and 3.2, we present a simple description of this set $\mathcal{S}$. Clearly $0 \in \mathcal{S}$ and $\mathcal{S} - \{0\}$ is an open set. It is easy to see that

$$\{\mu : 0 \leq \mu < E[(X - v + \mu)+]\} \subseteq \mathcal{S}$$

and the subset $\{\mu : 0 \leq \mu < E[(X - v + \mu)+]\}$ is of the form $[0, a)$ for some positive number $a$. In general, the set $\mathcal{S}$ might not necessary be an interval. Hence the second part of Prop 1.1 should be understood for relatively small $\mu$. If $\mu$ does not belong to $\mathcal{S}$, the existence of an optimal solution to Problem 1.2 as well as the Problem 1.1 is not obvious.

**The function $\mathcal{P}(\mu)$ is not necessary monotonic.** We present a numerical example to show that the function $\mathcal{P}(\mu)$ is not always monotonic with respect to the premium $\mu$. Henceforth, we justify our aforementioned approach to solving Problem 1.1. As an example, assume $W_0 = 1000$, $v = \frac{W_0}{100} = 10$, the loss $X$ is distributed with the following density function (triangle shape):

$$f(x) = \frac{2}{ab} x 1_{x \leq a} + \left(\frac{2}{b} + \frac{2}{b(b-a)}(a-x)\right) 1_{a < x \leq b} \quad ; \quad a = 25 \quad ; \quad b = 250.$$

The premium principle is $\mu = (1 + \rho)E[I(X)]$ where $\rho = 0.12$ is the loading factor. Then for each value of $\mu \in (0, v)$, we solve the quantile $q$ so that:

$$\frac{\mu}{1 + \rho} = E\left[(X - (v - \mu))^+ 1_{X \leq q}\right]$$

We finally compute the probability of $X$ to be more than $q$, which is $\mathcal{P}(\mu)$. As shown in Figure 2, the function $\mathcal{P}(\mu)$ is first decreasing and then increasing with respect to the

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7Same remarks apply to both Proposition 3.1 and Prop 3.2 below.
premium $\mu$. Clearly, the optimum $\min_{0 \leq \mu \leq \Delta} \mathcal{P}(\mu)$ is not necessary the premium $\Delta$. Here, the minimum is attained when the optimal premium $\mu^*$ is approximately equal to 74 while $\Delta = 100$ (see Figure 2).

![Figure 2: Probability $\mathcal{P}(\mu)$ w.r.t. $\mu$](image)

The shape $\mathcal{P}(\mu)$ with respect to $\mu$ has important economic implications. When higher premium $\mu$ is paid, it is tempting to expect it reduces the insolvency risk, thus the smaller $\mathcal{P}(\mu)$. The above example shows that this is not case in general. On the other hand, the minimum insolvency probability is obviously decreasing in Problem 1.1 when $\mu$ increases.

We now provide several reasons why the regulator’s risk measure constraint induces risk aversion to risk neutral insurers. First, in our model it is optimal for a risk neutral insurer in the absence of regulators, not to buy insurance. When there exists a regulation constraint, we give the optimal insurance design. Second, the insurance demand can be measured by the amount spent on insurance, which is the premium $\mu$. For realistic situations, the higher the premium the lower is the ruin probability (which can also be interpreted as the more risk averse, the higher the insurance demand). Thus a stronger control of regulators induces a higher reinsurance demand. The following reason is more appealing, which is illustrated by Figure 3. Figure 3 displays the trade-off between the expected return (through the final expected wealth) with respect to the risk (measured by the level $\alpha$). As we observe, this trade-off shape is concave which is similar to the trade-off between risk and return for a risk-averse investor. This figure clearly implies that risk neutral insurance companies react as risk-averse investors in the presence of regulators (this is an induced risk aversion$^8$).

There are other factors which motivate the risk aversion of risk neutral insurers. We provide a comparison later in section 4 of our approach with the previous contributions of Froot et al. (1993) and Caillaud et al. (2000).

$^8$Terminology proposed by Caillaud, Dionne and Jullien (2000) in another context.
2 Dual Problem

The objective in this section is to characterize the optimal indemnity that minimizes the premium while the VaR constraint is satisfied. It is the dual problem of Problem 1.1 and has been partly studied in the previous literature. This section is mainly used to clarify some technical points and present link with previous literature.

Problem 2.1 Solve the indemnity $I$ that minimizes the premium (minimum amount spent on reinsurance) while a compulsory VaR constraint is imposed.

$$\min_I (E[I(X) + C(I(X))]) \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ \Pr(L > v) \leq \alpha \end{cases}$$

When the premium is expressed as the actuarial value plus a loading factor), the dual problem also maximizes the expected final wealth subject to the compulsory VaR constraint.

However, it is not obvious to solve the dual problem 2.1 directly because the objective function $E[I(X)]$ is involved in the constraints of Problem 2.1. Precisely, the constraint $\Pr(L > v) \leq \alpha$ is equivalent to

$$\Pr(X - I(X) + E[I(X) + C(I(X))] > v) \leq \alpha$$

Thanks to Prop 1.1, we are able to derive one optimal indemnity $I(X)$ of the dual problem 2.1. The following proposition follows from Prop 1.1 naturally.

Proposition 2.1 Given a probability $\alpha_1 \leq \alpha$, $\Delta(\alpha_1)$ is the minimal solution of the premium equation (7) in which $q$ is replaced by the $(1-\alpha_1)$-quantile of $X$. Then the indemnity
\( I^{*}(X) \) solves for the Dual Problem 2.1, where
\[
\alpha^* = \arg\min_{[0,\alpha]} \Delta(\alpha_1). \quad (8)
\]

There are some subtle issues in investigating Problem 1.1 and its dual problem 2.1. These issues arise because both problems are not standard in Arrow-Raviv's expected utility paradigm in which either the objective function or the constraint is not concave. Moreover the constraints in both Problem 1.1 and Problem 2.1 are not necessary binding. We now present more details for this issue.

By Prop 1.1, \( P(\mu) \) represents the minimal probability of \( \{ L > v \} \) by purchasing insurance contracts via exactly premium amount \( \mu \). We have observed that, as a function of the premium \( \mu \), \( P(\mu) \) is not always decreasing. In other words, the premium constraint is not binding in Problem 1.1. On the other hand, since \( \Delta(\alpha) \) in the last proposition is the minimal premium when the insolvency probability \( \Pr(L > v) = \alpha \), the function \( \Delta(\alpha) \) is not necessary monotone by the same reason. Therefore, the constraint \( P(W < W_0 - v) \) is not binding for the optimal indemnity in Problem 2.1.

**Link to Previous Literature**

Wang et. al (2005) consider the following problem:

**Problem 2.2** \( \max_{I} \left( \mathbb{E}[W] \right) \) s.t. \[
\begin{align*}
0 \leq I(X) & \leq X \\
\Pr\{W > \mathbb{E}[W] - v\} & \geq 1 - \alpha
\end{align*}
\]

where the premium is \((1 + \rho)\mathbb{E}[I(X)]\). In what follows we first derive the solution of Problem 2.2 and then compare this solution with the solution derived in Prop 1.1. At last we point several serious issues in the solution presented in Wang et. al (2005).

The dual problem of Problem 2.2 is reduced to a sequence of the following problems:

**Problem 2.3** Solve the indemnity such that
\[
\min_{I(X)} \Pr\{W < \mathbb{E}[W] - v\} \quad \text{s.t.} \quad \begin{cases} 
0 \leq I(X) \leq X \\
\mathbb{E}[I(X)] = \Delta
\end{cases}
\]

By fixing the actuarial value of the indemnity \( I(x) \) as \( \Delta \), the event \( \{ W < \mathbb{E}[W] - v \} \) is the same as the event \( \{ I(X) < X - v - \mathbb{E}[X] + \Delta \} \). By replacing \( \Delta \) in the proof of Prop 1.1 by \( \Delta - \mathbb{E}[X] \) in the present situation.

**Proposition 2.2** Assuming \( X \) has a continuous cumulative distribution function over \( \mathbb{R}^+ \), and assume that \( \Delta < \mathbb{E}\left[ (X - v - \mathbb{E}[X] + \Delta)^+ \right] \). Then there exists a solution
\[
I(X; \Delta) = (X + \Delta - v - \mathbb{E}[X])^+ \mathbb{1}_{v - \Delta + \mathbb{E}[X] \leq X \leq v - \Delta + \mathbb{E}[X] + \frac{1}{\lambda}} \quad (9)
\]
of Problem 2.3, where $\lambda$ is determined such that: $E[I(X; \Delta)] = \Delta$. The solution to the dual problem of Problem 2.2 writes as $I(X; \Delta_1^*)$ where $\Delta_1^*$ solves the static minimization problem

$$\min_{0 \leq \Delta_1 \leq \Delta} \Pr \left\{ X > v - \Delta + E[X] + \frac{1}{\lambda} \right\}. \tag{10}$$

To explain the Proposition 2.2, some remarks are given in order.

**Remark 2.1**

By Prop 2.2, the indemnity $I(X; \Delta)$ doesn’t depend on the premium of the indemnity, because the premium has been canceled out from the difference $W - E[W]$. The VaR constraint is expressed as $\Pr\{W > E[W] - v\} \geq 1 - \alpha$. It is indeed independent of costs (of the loading factor $\rho$) because $W - E[W] = W_0 - X + I - (1 + \rho)E[I] - E[W_0 - X + I - (1 + \rho)E[I]] = (I - E[I]) - (X - E[X])$. Therefore, their result is not intuitive since the minimal insolvency probability of Problem 2.3 is independent of the premium of the indemnity contract. This unattractive feature of the optimal indemnity design comes from the probability $\Pr\{W < E[W] - v\}$, which is not the VaR concept usually used in the industry. In fact, the event $\{W < E[W] - v\}$ depends mainly on the volatility of $W$ (for instance, for normal distributed $W$), instead of the worst scenario of $W$.

**Remark 2.2**

Even though Problem 2.2 misses the economic insight by the previous remark, the original Problem 2.3 can be solved by using Prop 2.2 from a technical perspective. In fact, the optimal (minimal) actuarial value is solved by

$$\Delta^* := \min \left\{ \Delta : \Pr \left\{ X > v - \Delta + E[X] + \frac{1}{\lambda} \right\} \leq \alpha \right\}$$

and the optimal indemnity of Problem 2.2 is given by $I(X; \Delta^*)$. However, as mentioned earlier, this design doesn’t reflect the trade-off between the premium and the insolvency probability. Therefore, this insurance design is not useful for risk management purpose.

**Remark 2.3**

There is another critical point in Wang et al. (2005). Similarly to the function $P(\Delta)$, the function $\Pr\{X > v - \Delta + E[X] + \frac{1}{\lambda}\}$ is not increasing with the actuarial value $\Delta$. Therefore, the for the optimal indemnity the probability constraint $\Pr\{W < E[W] - v\}$ is not necessary binding. This point is clearly missing in Wang et al. (2005).\(^9\)

\(^9\)See Wang et al. (2005) Page 164. They claimed that “the probability constraint $P(W < E[W] - v)$ must be binding for optimality, since $X$ has a continuous probability distribution.” Moreover, they claimed that, at the bottom of Page 164, “$\Pr\{X - I(X) > v + E[X] - E[I]\}$ is increasing in $E[I(X)]$.”
3 Optimal Reinsurance under Tail Risk Measures

We have explored the optimal reinsurance contract under VaR constraint. Adopting an optimal reinsurance arrangement under VaR constraint, insurance companies choose to leave the worst states uninsured (see Figure 1). This result is reasonable giving the maximum expected wealth objective, since those worst states are in fact the most expensive ones to insure against. Indeed, such contracts reduce the probability to incur large loss but they do not limit the losses’ amount in worst states. Thus VaR requirements lead companies to ignore losses in the tail of the distribution. A deeper analyze of the consequences of Value-at-Risk management in the financial market can be found in Basak and Shapiro (2001). They show that VaR risk managers often choose larger risk exposure to risky assets and consequently incur larger losses when losses occur.

In this section we discuss optimal reinsurance when different risk measures are imposed. For instance, the Conditional Tail Expectation (CTE) risk measure has been studied extensively in theory and is used by Canadian regulation in practice (for some index linked insurance annuities). The motivation of CTE is to limit the risk exposure toward large losses instead of the probability of bad states only. We show that, even though its theoretical advantage (see Artzner et al. (1999), and Basak and Shapiro (2001)), the CTE constraint implies the same optimal design as in the last section. Then we consider another risk measure, which is linked to the second moment of the excess of loss. We prove that the deductible policy is optimal when the second risk measure constraint is satisfied.

For simplicity of notations we assume the premium is a function of the actuarial value of the indemnity throughout this section which is consistent with the previous work by Raviv (1979). For instance, this includes a linear premium principle $\mu = (1 + \rho)E[I(X)]$, where $\rho$ is a constant loading factor.

3.1 CTE Risk Measure

The optimal design under the conditional tail expectation can be stated as follows.

Problem 3.1 Solve the indemnity such that

$$
\min \left\{ \mathbb{E} \left[ \left( W_0 - W \right) I_{W_0 - W > v} \right] \right\} \quad \text{s.t.} \quad \begin{cases} 0 \leq I(X) \leq X \\ \mathbb{E}[I(X)] \leq \Delta \end{cases}
$$

where the loss level $v$ is exogenously specified. We point out that this framework is consistent with the one investigated in Basak and Shapiro (2001), where $v$ is not related to $(1 - \alpha)$-quantile of the loss. For the linear premium principle, the constraint $\mathbb{E}[I(X)] \leq \Delta$ equivalents to that the premium is bounded by $(1 + \rho)\Delta$. Hence, Problem 3.1 solves the optimal reinsurance contract with minimum loss exposure on the loss states $\{W_0 - W > v\}$ by paying at most premium $(1 + \rho)\Delta$.

Similar to the previous discussions for the VaR risk measure, the solution of Problem 3.1 also involves two steps. We first solve Problem 3.1 by fixing the premium, reducing it to Problem 3.2 below.
Problem 3.2 Solve the indemnity such that

\[
\min_{I} \{ \mathbb{E}[(W_0 - W) I_{W_0 - W > v}] \} \quad s.t. \begin{cases} 0 \leq I(X) \leq X \\ \mathbb{E}[I(X)] = \Delta \end{cases}
\]

The solution of Problem 3.2 is given by:

**Proposition 3.1** Assuming \( X \) has a continuous cumulative distribution function strictly increasing on \([0, +\infty)\). Assume that \( \Delta \in \mathcal{S} \), then there exists a solution

\[
I_{\Delta}^c(X) = (X + (1 + \rho)\Delta - v)^+ I_{\nu - (1 + \rho)\Delta \leq X \leq \nu - (1 + \rho)\Delta + \frac{\Delta}{\rho}}
\]

of Problem 3.2, where \( \lambda > 0 \) satisfies that \( \mathbb{E}[I_{\Delta}^c(X)] = \Delta \). Let \( W_{\Delta}^c \) be the final wealth derived from \( I_{\Delta}^c(X) \). Then the indemnity \( I_{\Delta}^c(X) \) solves Problem 3.1, when \( \Delta^* \) minimizes:

\[
\min_{0 \leq \Delta_1 \leq \Delta} \mathbb{E}[(W_0 - W_{\Delta_1}^c) I_{W_0 - W_{\Delta_1}^c > v}]
\]

Proposition 3.1 shows that a risk neutral insurer behaves risk averse in the presence of CTE risk measure constraint since it is optimal to buy some reinsurance. Moreover, insurers have no incentives to protect themselves against large losses under the conditional tail expectation’s constraint. This proposition derives the same adverse incentive for the insurer as in the VaR case. At first sight, this result seems surprising because people often argue that CTE risk measure works better than the VaR risk measure (see Basak and Shapiro (2001), Artzner et al. (1999)). In fact, this result is still intuitive. Write \( L = W_0 - W \). In terms of the loss variable \( L \), Problem 3.2 is to solve for

\[
\min_{L} \mathbb{E}[L I_{L > v}]
\]

subject to \( \mathbb{E}[L] \) is fixed, and \( \mu \leq L \leq \mu + X \). Therefore, the objective is to take consideration of the tradeoff between the amount \( L \) and the probability \( P_r\{L > v\} \). If the indemnity is large on “bad” states (when \( X \) is large), then the loss \( L \) is small on the “bad” states. However, because \( \mathbb{E}[L] \) is fixed, then there might have large loss on “good” states. On the other hand, if the indemnity is small on “bad” states, then loss is large on “bad” states. Consequently, there exists small loss on “good” states. Because of the risk aversion of the insured, the optimal indemnity is to have small loss on “good” states, and henceforth large loss on “bad” states.

Both Proposition 1.1 and 3.1 are consistent with the empirical findings of Froot (2001). Froot (2001) find that most insurers purchase relatively little reinsurance against catastrophes’ risk. He also provides a number of possible reasons for these departures from theory. In the presence of a constraint on the risk (through VaR or CTE), this pattern of the reinsurance contract profile is preserved.

From the design perspective, the reinsurance contract with little indemnity in case of a large loss event is not desirable for regulators (who want to protect insurers from a too high risk exposure), moreover it incurs moral hazard for reinsurers. Indeed, reinsurers
need costly investigation to reveal correctly loss information from the insurer, because insurer don’t insured on the large loss state. By increasing the premium to cover the verification cost from reinsurer’s side, it makes even worse for the insurer. Therefore either VaR or CTE constraints are not enough to incite companies to purchase insurance against catastrophic (large loss) risks.

In the next subsection, however, we show that, strong risk control requirement by regulators provide reinsurance purchase incentives for large loss risk. In fact, deductible policies are optimal for the insurer when alternative risk measures are considered.

3.2 Emphasize the Right Tail Distribution

The risk measure we consider in this subsection is based on the expected square of the excessive loss. This risk measure is related to the variance tail measure and thus useful when the variability of the loss is high\(^\text{(10)}\). Precisely, we study the following problem for insurer.

**Problem 3.3** Find the optimal indemnity \(I\) that solves:

\[
\min_I \{ E \left[ \left( W_0 - W - v \right)^2 \mathbb{I}_{W_0 - W > v} \right] \} \quad \text{s.t.} \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] \leq \Delta \end{cases}
\]

The objective of Problem 3.3 is to minimize the square of excess loss. Comparing with CTE, this risk measure \(E[\left( W_0 - W - v \right)^2 \mathbb{I}_{W_0 - W > v}]\) pays more attention on the loss amount over the loss states \(\{W_0 - W > v\}\). Then, it is termed as “expected square of excessive loss measure”.\(^\text{(11)}\) The Proposition 3.2 below solves the Problem 3.3. We first write:

**Problem 3.4** Find the optimal indemnity that solves:

\[
\min_I \{ E \left[ \left( W_0 - W - v \right)^2 \mathbb{I}_{W_0 - W > v} \right] \} \quad \text{s.t.} \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] = \Delta \end{cases}
\]

**Proposition 3.2** Assuming \(X\) has a continuous cumulative distribution function strictly increasing on \([0, +\infty)\). Assume that \(\Delta \in S\). Then the solution of Problem 3.4 is a deductible indemnity \((X - d)^+\),

where the deductible level \(d = d_\Delta\) is determined by the actuarial value \(\Delta\) of the indemnity. Let \(W_\Delta\) be the final wealth when the reinsurance indemnity is given by \((X - d)^+\). The solution to Problem 3.3 is a deductible indemnity \((X - d^*)^+\) where \(d^*\) solves the following minimization problem:

\[
\min_{0 \leq \Delta_1 \leq \Delta} E \left[ \left( W_0 - W_\Delta - v \right)^2 \mathbb{I}_{W_0 - W_\Delta > v} \right]
\]

\(^{(14)}\)

\(^{10}\)For more details on this risk measure we refer to Furman and Landsman (2007).

\(^{11}\)This measure is not a coherent risk measure in the sense of Artzner et al. (1999) though.
Proposition 3.2 states that deductibles are optimal when a constraint on the expected square of excessive loss is imposed. In term of \( L = W_0 - W \). This problem is subject to
\[
\min \mathbb{E}[L^2 1_{L>v}]
\]
subject to \( \mathbb{E}[L] \) is fixed and \( \mu \leq L \leq \mu + X \). In contrast with Problem 3.1, the objective function in Problem 3.4 involves \( L^2 \) which dominates the constraint \( \mathbb{E}[L] \). Then, intuitively, the optimal indemnity should minimize the loss \( W_0 - W \) over the bad states as small as possible. Hence the optimal indemnity is deductible. Proposition 3.2 justifies this intuition.

By Proposition 1.1, 3.1 and 3.2 all together, we see that stronger regulatory control on the risk exposure leads to more efficient risk-sharing between insurer and reinsurer. It is worth to mention the methodology behind these propositions. To study the usefulness of a regulatory risk measure, understanding its implications on the market is important. Optimal insurance under risk measures is often investigated independently of the effects on the market. Proposition 1.1, 3.1 and 3.2 display the market effect on the risk measures by investigating the profile of the optimal reinsurance contract.

4 Optimal Indemnity with Financing Imperfections

We have shown that risk neutral insurers behave risk averse because of the enforcement of risk measure constraints. The profile of the optimal reinsurance contract depend on how the risk control policy is requested and implemented. Other factors, mentioned earlier in the literature, contribute to the risk averse attitude of risk neutral insurance companies. In this section, we compare our approach with two alternatives to generate the reinsurance demand by Froot, Scharfstein and Stein (1993) and Caillaud, Dionne and Jullien (2000). Both show that external financing generates an insurance demand by risk neutral firms. This amounts to comparing effects of enforcement regulation constraints and voluntary risk management.\(^{12}\)

Froot, Scharfstein and Stein (1993) consider a value-maximizing insured facing financing imperfections increasing the cost of the raising of external funds. The imperfections include cost of financial distress, taxes, managerial motives or other capital market imperfections. These imperfections alter the shape of the value function \( P(W) \) of the firm where \( W \) denotes the internal capital. Under a fairly general condition of the loss \( X \), Froot, Scharfstein and Stein (1993) prove that the partial differentiates \( P_{WW} < 0, P_W \geq 1 \) by using a costly-state verification model of external financing (See Townsend (1979)). Purchasing a reinsurance contract \( I(X) \) by paying the premium \( \mu \), the final wealth writes as: \( W = W_0 - X + I(X) - \mu \). The insurer makes a reinsurance decision by maximizing the expected value of the firm \( \mathbb{E}[P(W)] \). Therefore, the risk neutral firm behaves like a risk averse individual with concave utility function \( P(\cdot) \). It has been showed by Arrow (1963)

\(^{12}\)Firms makes financing decision and insurance decision to increase firm value. Hence the risk management policy is voluntary comparing with the compulsory risk management requirement imposed by regulator.
that deductible indemnity is optimal for a risk averse individual. Hence, the optimal reinsurance contract in Froot, Scharfstein and Stein (1993)’s framework is a deductible (no insurance for small losses, full insurance above the deductible level).

Caillaud, Dionne and Jullien (2000) examine the problem from a different angle by rationalizing the use of insurance covenants in financial contracts, say corporate debts. In Caillaud, Dionne and Jullien (2000), external funding for a risky project can be affected by an accident during its realization. Since accident losses and final returns are private information and can be costly evaluated by outside investors, the optimal financial contract must be a bundle of a standard debt contract and an insurance contract which involves full coverage above a straight deductible. Hence, small loss is not insured because of the auditing costs and the bankruptcy costs.

Figure 4: Comparison of reinsurance contracts.
Reinsurance indemnity \( I \) w.r.t. the loss \( X \)

Figure 4 displays the optimal insurance contract based on either regulatory constraints or costly external funding. On Figure 4, the two indemnities have the same actuarial value, thus a similar premium. The truncated deductible indemnity is optimal for VaR or CTE constraints, while the deductible indemnity is optimal for either the square of the expected loss risk measure or voluntary risk management.

We compare the risk measure constraint and the voluntary risk management policy. In all cases, small losses stay uninsured (that reduces costs and moral hazard). But there are significant differences on the medium loss and large loss. A Value-at-Risk constraint or a CTE constraint is not enough to induce insurers to protect themselves against large loss amounts. The probability of having a large loss is controlled but the amount of the loss is not. This is the opposite to the deductible indemnity. In the latter case, companies protect large losses up to a fixed amount (deductible), controlling also their default risk. Strong risk control such as the square of the expected loss risk measure provide incentive to insured large loss, hence the optimal indemnity under this risk measure is identical with the one under voluntary risk management policy.

Regulatory requirement and firm’s risk management policy lead to different protection (or hedging) strategy. VaR and CTE risk management policies provide a better protection
on medium losses. If the company only implement the enforced constraint without doing a risk averse risk management, it will benefit on average until a large loss occurs. On the other hand, the firm’s risk management policy focuses on the large loss but this “over hedge” strategy on the large loss reduces the gain on the medium loss level. Thus the enforcement of VaR and CTE regulations will be efficient only in the presence of firm’s risk management program (such as the one suggested by Froot et al. (1993) or Caillaud et al. (2000)).

We now move to the reinsurance contracts in the marketplace.

5 Reinsurance and Capital Market

In this section, we first compare our results to traditional reinsurance policies, then interpret reinsurance arrangements as a derivatives portfolio written on a loss index.

We find that the optimal reinsurance contract is not available in the reinsurance market due to moral hazard issues. However, the optimal strategy under VaR can possibly be implemented in the capital market, as soon as a reference index strongly correlated to the insurer’s loss is traded.

5.1 Comparing Existing Reinsurance Contracts and our Results

Froot (2001) underlines that most reinsurance arrangements are “excess-of-loss layer” with a retention level (the deductible level that losses must exceed before coverage is triggered), a limit (the maximum amount reimbursed by the reinsurer) and an exceeding probability (probability losses are above the limit). The contract writes as:

\[ I_1(X) = (X - d) \mathbb{1}_{X \in [d, l]} + (l - d) \mathbb{1}_{X > l}. \]  

(15)

where \( d \) is the deductible level, \( l \) stands for the upper limit of the coverage, thus \( l - d \) is the maximum indemnity. This contract is a popular contract involving a stop loss rule (that is a deductible) with an upper limit on coverage.

Let us compare it with our previous results. We already pointed out that the optimal design might induce moral hazard, in particular insurers have incentive to partly hide their losses. A contract such as the one given by expression (15) does not have this drawback since the indemnity is non-decreasing with the loss amount. Figures 5 illustrates a comparison of these two designs with arbitrary parameters.

We compare the optimal contract and the deductible with an upper limit. We assume the premium \( \mu \) is the same in both contracts. Thus, if one contract reimburses more than another one for one range of loss amounts, then it must be the contrary for another range.

---

\[^{13}\] The optimal insurance contract design combining regulatory constraints and costly external funding cost (that is for a risk averse firm) is beyond the scope of this paper.

\[^{14}\] Policies with upper limit on coverage could be derived from minimizing some risk measures under a mean variance premium principle. They also have been extensively analyzed by Cummins and Mahul (2004).
of losses amounts. On Figure 5, the plain line corresponds to the optimal contract under a VaR constraint and the dash line is the capped contract. They have the same premium $\mu$, so the coverage provided by the optimal contract is better for medium losses but worse for extreme losses.

Assuming that the company cannot manipulate the actual amount of loss, the truncated deductible is then a possible design and seems to be a better design. For a similar amount of money spent in reinsurance, the company obtains a better coverage for medium losses which are the most possible losses but a lower for extreme losses. Moreover, the coverage provided by the upper-limit contract is fixed in the case when an extreme loss occurs (equal to the level of the upper-limit). Thus, such a contract does not avoid bankruptcy.

Moral hazard can be also reduced by a coinsurance treaty, where the coverage is partly provided by the reinsurer (for instance $\theta = 95\%$) (see Cummins, Lalonde and Phillips (2004))

$$I_\theta(X) = \theta(X - d)1_{X \in [d,l]} + \theta(l - d)1_{X > l}, \quad \theta < 1.$$  

The recent work of Froot (2001) gives an overview of the market for catastrophe risk. He notices that “most insurers purchase relatively little cat reinsurance against large events and that premiums are high relative to expected losses”. He explains that both reinsurance and CAT bonds generally trade at significant margins above the expected loss and that insurers tend to retain rather than share their large event risks. This is in concordance with our study. Truncated deductible are indeed optimal when risk constraints are linked to a VaR or a CTE constraint. Then insurance companies choose to let uninsured the worst possible states.

### 5.2 Alternative on Capital Market

Recently, it has become widely appreciated that a single natural hazard could result in damages of several billions. For instance the total insured US catastrophic losses for 2005
are estimated to be more than $50 billion, where the three major hurricanes Katrina, Rita and Wilma make up 90% of the total loss of the year\textsuperscript{15}. Even if the insurance industry’s equity capital would be enough to absorb lots of catastrophic events, Cummins, Doherty and Lo (2002) explain that many insurers can become insolvent depending on the distribution of damage and their portfolio of policies. They note that in absence of costs, the Pareto optimal way to share risks is to mutualize all risks between all insurers.

The traditional instrument to spread risks between insurers is reinsurance. By reinsuring a layer of one line of business or of a specific risk, insurers buy and sell options on the loss index. Assuming risks can be measured in terms of an index (for instance, temperature, a specified event or wind speed), then a reinsurance arrangement can be interpreted as a portfolio of derivatives written on this underlying index.

More precisely, Froot (2001) and Cummins, Lalonde and Phillips (2004) compare reinsurance layers with call spreads. They explain how insurers hedge their risks by forming a portfolio consisting in its losses (assume to be traded as a loss index \( X \)) and a position in call option spreads on the loss index,

\[
X - I_\theta(X) = X - \theta [(X - d)^+ - (X - l)^+] \tag{16}
\]

where \( \theta \) corresponds to a coinsurance treaty. The indemnity is:

\[
I_\theta(X) = \begin{cases} 
0 & \text{If } X < d \\
\theta(X - d) & \text{If } X \in [d, l] \\
\theta(l - d) & \text{If } X > l 
\end{cases}
\]

In presence of regulators’ minimum capital requirement and VaR or Conditional Tail expectation maximum risk exposure, we show that the optimal reinsurance arrangement is:

\[
I(X) = (X - d) \mathbb{1}_{X \in [d, q]}
\]

where \( d \) is the deductible level and \( q \) the upper limit. Then,

\[
I(X) = (X - d)^+ - (X - q)^+ - (q - d) \mathbb{1}_{X > q}
\]

which corresponds to a portfolio of derivatives, a long position on a call and a short position on a put and on a barrier bond (activated when the underlying \( X \) is above \( q \)). Equivalently, the company possesses:

\[
X - I(X) = X - [(X - d)^+ - (X - q)^+ - (q - d) \mathbb{1}_{X > q}] .
\]

Comparing this portfolio with the call spread (16), we show it is optimal for companies to sell the bond corresponding to the right tail risk. In some sense, they optimally give up the right tail, because there is no reason to pay for (useless) reinsurance after being ruined (when the upper-limit on coverage is not enough to stay solvent).

\textsuperscript{15}Source: The annual report Guy Carpenter, “US Reinsurance Renewals at January 1, 2006”
Using a simulation model, Cummins, Lalonde and Phillips (2004) study the efficiency of hedging with reinsurance or index-linked securities. They explain that reinsurance contracts are sold over the expected loss and that it is less efficient than hedging using contracts actuarially fairly priced. Insurance-linked securities are mostly competitive with reinsurance in terms of price and hedge efficiency since they can be traded at significantly lower margins. Duplicate reinsurance arrangements on the market is a worth alternative risk transfer.

Conclusions

New risk management programs are currently being implemented to the insurance market (with Solvency II project for example). We focus on risk-neutral insurance companies subject to several tail risk measures imposed by regulators. We derive the design of the optimal reinsurance contract to maximize the profit, or equivalently to minimize the premium, when the regulatory constraints are satisfied. We show that insurance companies have no incentives to protect themselves against extreme losses when regulatory requirements are based on Value-at-Risk or Conditional Tail Expectation maximum risk exposure. These results confirm observed behaviors of insurance companies that prefer to let partly uninsured the high layers of risks (Froot (2001)). Further, we show that an alternative risk management measure would lead insurance companies to fully hedge the right tail of the loss distribution.

In this paper we make model assumptions that there are no transaction cost for issuing and purchasing reinsurance contracts, no background risk, and a single loss during the period of insurance protection. Moreover both issuer and issued are risk neutral, both parties know the probability distribution of the loss. Even though of the above mentioned model limitations, the results of this paper could be still used as “prototypes” by insurance companies to design optimal risk management strategies, as well as by regulators to impose appropriate risk measures. Because of the link and similarity between reinsurance market and capital market, our results also present alternative risk transfers mechanisms in the capital market.
A Proofs

Recall that final wealth $W$ is given by $W = W_0 - \mu - X + I(X)$. Then, the event \( \{W \geq W_0 - v\} \) is equivalent to, in terms of the coverage $I(X)$,\[
\{I(X) \geq \mu + X - v\}
\]

A.1 Proposition 1.1

The problem 1.2 could be reformulated as follows.
\[
\max_I \Pr (I(X) \geq \mu + X - v) \quad \text{s.t.} \quad \left\{ \begin{array}{l}
0 \leq I(X) \leq X \\
\mathbb{E}[I(X) + C(I(X))] = \mu
\end{array} \right.
\]

Lemma A.1 If $Y^*$ satisfies the three following properties:
(i) $0 \leq Y^* \leq X$,
(ii) $\mathbb{E}[Y^* + C(Y^*)] = \mu$,
(iii) There exists a positive $\lambda > 0$ such that for each $\omega \in \Omega$, $Y^*(\omega)$ is a solution of the following optimization problem:
\[
\max_{Y \in [0, X]} \left\{ I_{\mu+X(\omega)-v \leq Y} - \lambda(Y + C(Y)) \right\}
\]
then $Y^*$ solves the current optimization problem 1.2.

Proof. Given a coverage $I$ which satisfies the constraints of the optimization program 1.2. Therefore, using (iii), we have,
\[
\forall \omega \in \Omega, \quad I_{\mu+X(\omega)-v \leq Y^*} - \lambda(Y^* + C(Y^*)) \geq I_{\mu+X(\omega)-v \leq I(\omega)} - \lambda(I(\omega) + C(I(\omega)))
\]
Thus,
\[
I_{\mu+X(\omega)-v \leq Y^*(\omega)} - I_{\mu+X(\omega)-v \leq I(\omega)} \geq \lambda(Y^*(\omega) + C(Y^*(\omega)) - I(\omega) - C(I(\omega)))
\]
We now take the expectation of this inequality. Therefore by condition (ii) one obtains,
\[
P(\mu + X - v \leq Y^*) - P(\mu + X - v \leq I) \geq \lambda(\mu - \mathbb{E}[I + C(I)])
\]
Therefore, applying the constraints of the variable $I$, $\mathbb{E}[I(X) + C(I(X))] = \mu$,
\[
P(\mu + X - Y^* \geq v) - P(\mu + X - I \geq v) \geq 0
\]
The proof of this lemma is completed. \qed
Lemma A.2 When $\mu \leq v$, each member of the following family $\{Y_\lambda\}_{\lambda > 0}$ satisfies the conditions (i) and (iii) of lemma A.1.

$$Y_\lambda(\omega) = \begin{cases} 0 & \text{if } X(\omega) < v - \mu \\ X(\omega) + \mu - v & \text{if } v - \mu \leq X(\omega) < v - \mu + D\left(\frac{1}{\lambda}\right) \\ 0 & \text{if } X(\omega) > v - \mu + D\left(\frac{1}{\lambda}\right) \end{cases}$$

where $D$ is the inverse of $y \to y + C(y)$.

**Proof.** The property (i) is obviously satisfied. Indeed we only study the case when $v$ is more than the premium $\mu$, that is $v \geq \mu$.

First, if $X(\omega) < v - \mu$, then $\mu + X(\omega) - v < 0$, the function to maximize over $[0, X(\omega)]$ is equal to $1 - \lambda(Y + C(Y))$, decreasing over the interval $[0, X(\omega)]$ (since $C' > -1$), the maximum is thus obtained at $Y^*(\omega) = 0$.

Otherwise, $X(\omega) \geq v - \mu$. Since $\mu \leq v$, one has $\mu + X(\omega) - v \leq X(\omega)$. We consider two cases: firstly, if $Y \in [0, \mu + X(\omega) - v)$, then the function to maximize is $-\lambda(Y + C(Y))$. It is decreasing with respect to the variable $Y$. Its maximum is 0, obtained at $Y = 0$. Secondly, if $Y \in [\mu + X(\omega) - v, X(\omega)]$, then the function to maximize is $1 - \lambda(Y + C(Y))$. It is decreasing. Its maximum is obtained at $Y = \mu + X(\omega) - v$ and its value is $1 - \lambda(\mu + X(\omega) - v + C(\mu + X(\omega) - v))$. We compare the value $1 - \lambda(\mu + X(\omega) - v + C(\mu + X(\omega) - v))$ and 0 to decide whether the maximum is attained at $Y = X(\omega) + \mu - v$ or $Y = 0$.

$$1 - \lambda(\mu + X(\omega) - v + C(\mu + X(\omega) - v)) \geq 0 \iff \mu + X(\omega) - v + C(\mu + X(\omega) - v) \leq \frac{1}{\lambda}$$

Let $D = (Y + C(Y))^{-1}$ that exists since $Y + C(Y)$ is increasing, then:

$$X(\omega) \leq v - \mu + D\left(\frac{1}{\lambda}\right)$$

Lemma A.2 is proved. \(\square\)

**Proof of Proposition 1.1** Thanks to lemmas A.1 and A.2, it suffices to prove that there exists $\lambda > 0$ such that $Y_\lambda$ defined in lemma A.2 satisfies the condition (ii) of lemma A.1. We then compute its associated cost function.

$$\mathcal{E}_\lambda := \mathbb{E}\left[ (X + \mu - v) 1_{X \in [v-\mu,v-\mu+D(\frac{1}{\lambda})]} + C\left((X + \mu - v) 1_{X \in [v-\mu,v-\mu+D(\frac{1}{\lambda})]}\right)\right]$$

It is obvious then:

$$\lim_{\lambda \to 0^+} \mathcal{E}_\lambda = \mathbb{E}\left[ (X - v + \mu) + C\left((X - v + \mu)\right)\right], \quad \lim_{\lambda \to +\infty} \mathcal{E}_\lambda = 0.$$

By Lebesgue dominance theorem we can easily prove the convergence property of $\mathcal{E}_\lambda$ with respect to the parameter $\lambda$. Then the existence of a solution $\lambda^*_\mu \in \mathbb{R}_+^*$ such that $\mathcal{E}_\lambda = \mu$
comes from the assumption on the continuous distribution of $X$ and thus the continuity of $E_\lambda$. Thus we have proved the first part of this Proposition. The second part follows easily from the first part.

A.2 Proposition 3.1

Assuming $\mu = g(E[I(X)])$, Problem 3.1 could be rewritten as:

$$\min_I \left( E[(\mu + X - I)1_{\mu+X - I > v}] \right) \quad s.t. \quad \begin{cases} 0 \leq I(X) \leq X \\ E[I(X)] = \Delta \end{cases}$$

Lemma A.3 If $Y^*$ satisfies the three following properties:

(i) $0 \leq Y^* \leq X$,
(ii) $E[Y^*] = \Delta$,
(iii) There exists a positive $\lambda > 1$ such that for each $\omega \in \Omega$, $Y^*(\omega)$ is a solution of the following optimization problem:

$$\min_{Y \in [0, X(\omega)]} \left\{ (\mu + X(\omega) - Y) 1_{Y < \mu + X(\omega) - v} + \lambda Y \right\}$$

then $Y^*$ solves Problem 3.1.

Proof. The proof of lemma A.3 is similar to the proof of lemma A.1. □

Proof of Proposition 3.1: We use lemma A.3 and show that for $\lambda > 1$,

$$Y_\lambda(\omega) = \begin{cases} 0 & \text{if } X(\omega) < v - \mu \\ X(\omega) + \mu - v & \text{if } v - \mu \leq X(\omega) \leq v - \mu + \frac{v}{X - 1} \\ 0 & \text{if } X(\omega) > v - \mu + \frac{v}{X - 1} \end{cases}$$

satisfies conditions (i) and (iii) of lemma A.3. If $X(\omega) + \mu - v < 0$ then $Y^* = 0$. Otherwise $0 \leq \mu + X(\omega) - v < X$. Similar to the proof of Proposition 1.1 we can prove that $Y = \mu + X(\omega) - v$ is the maximum one if $v - \mu \leq X(\omega) \leq v - \mu + \frac{v}{X - 1}$, else the maximum one is $Y = 0$ if $X(\omega) > v - \mu + \frac{v}{X - 1}$.

We then compute its expectation.

$$E_\lambda := E \left[ (X + \mu - v) 1_{X \in (v - \mu, v - \mu + \frac{v}{X - 1})} \right]$$

It is obvious then:

$$\lim_{\lambda \to 1^+} E_\lambda = E \left[ (X - v + \mu)^+ \right], \quad \lim_{\lambda \to +\infty} E_\lambda = 0.$$ 

The existence of a solution $\lambda^* > 1$ such that $E_\lambda = \Delta$ comes from the assumption on the continuous distribution of $X$ and thus the continuity of $E_\lambda$. Thus we have proved the first part of Proposition 3.1. The second part follows easily from the first part. □
A.3 Proposition 3.2

Lemma A.4 If $Y^*$ satisfies the three following properties:
(i) $0 \leq Y^* \leq X$,
(ii) $\mathbb{E}[Y^*] = \Delta$,
(iii) There exists a positive $\lambda > 0$ such that for each $\omega \in \Omega$, $Y^*(\omega)$ is a solution of the following optimization problem:

$$\min_{Y \in [0, X(\omega)]} \{ (\mu + X(\omega) - Y - v)^2 1_{Y < \mu + X(\omega) - v} + \lambda Y \}$$

then $Y^*$ solves Problem 3.4.

Proof. Let $Y^*$ be a random variable satisfying the three above conditions of the lemma. On the other hand, given another available payoff $I$ which satisfies the constraints of the above optimization program. Therefore, using (iii), we have, $\forall \omega \in \Omega$,

$$\begin{align*}
(\mu + X(\omega) - Y^*(\omega) - v)^2 1_{Y^*(\omega) < \mu + X(\omega) - v} + \lambda Y^*(\omega) & \leq \\
(\mu + X(\omega) - I(\omega) - v)^2 1_{I(\omega) < \mu + X(\omega) - v} + \lambda I(\omega)
\end{align*}$$

Thus,

$$\begin{align*}
(\mu + X(\omega) - Y^*(\omega) - v)^2 1_{Y^*(\omega) < \mu + X(\omega) - v} - (\mu + X(\omega) - I(\omega) - v)^2 1_{I(\omega) < \mu + X(\omega) - v} & \leq \\
\lambda (I(\omega) - Y^*(\omega))
\end{align*}$$

We now take the expectation of the above inequality, therefore by condition (ii) one obtains,

$$\mathbb{E} [(\mu + X - Y^* - v)^2 1_{Y^* < \mu + X - v}] - \mathbb{E} [(\mu + X - I - v)^2 1_{I < \mu + X - v}] \leq \lambda (\mathbb{E}[I] - \Delta)$$

Therefore, applying the constraints of the variable $I$, $\mathbb{E}[I(X)] = \Delta$;

$$\mathbb{E} [(\mu + X - Y^* - v)^2 1_{Y^* < \mu + X - v}] \leq \mathbb{E} [(\mu + X - I - v)^2 1_{I < \mu + X - v}]$$

The proof of this lemma is completed. \qed

Lemma A.5 When $\mu \leq v$, each member of the following family \{${Y_\lambda}$\}$\lambda > 0$ satisfies the conditions (i) and (iii) of lemma A.4.

$$Y_\lambda(\omega) = \begin{cases} 
0 & \text{if } X(\omega) < v - \mu + \frac{\lambda}{2} \\
X(\omega) + \mu - v - \frac{\lambda}{2} & \text{if } v - \mu + \frac{\lambda}{2} \leq X(\omega)
\end{cases}$$

Proof. The property (i) is obviously satisfied. We now prove the property (iii).
First, if \( X(\omega) < v - \mu \), then \( \mu + X(\omega) - v < 0 \), the function to minimize over \([0, X(\omega)]\) is equal to \( \lambda Y \), increasing over the interval \([0, X(\omega)]\), the minimum is thus obtained at \( Y^*(\omega) = 0 \).

Otherwise, \( X(\omega) \geq v - \mu \). Since \( \mu \leq v \), one has \( \mu + X(\omega) - v \leq X(\omega) \). Thus we have to solve the optimization program under the assumption \( 0 \leq \mu + X(\omega) - v \leq X(\omega) \).

There are two cases: firstly, if \( Y \in \left[0, \mu + X(\omega) - v \right) \), then the function to minimize is

\[
\phi_1(Y) = (\mu + X(\omega) - v - Y)^2 + \lambda Y.
\]

Its minimum is \( \max \left(0, X(\omega) + \mu - v - \frac{\lambda}{2} \right) \). Secondly, if \( Y \in [\mu + X(\omega) - v, X(\omega)] \), then the function to minimize is

\[
\phi_2(Y) = \lambda Y.
\]

Its minimum is obtained at \( Y = \mu + X(\omega) - v \) and its value is \( \lambda (\mu + X(\omega) - v) \). We then compare this value with the previous minimum:

- When \( 0 < X(\omega) + \mu - v - \frac{\lambda}{2} \), \( \Phi_1(X(\omega) + \mu - v - \frac{\lambda}{2}) = \Phi_2(\mu + X(\omega) - v) - \frac{\lambda^2}{4} < \Phi_2(\mu + X(\omega) - v) \).

- When \( 0 > X(\omega) + \mu - v - \frac{\lambda}{2} \), \( \Phi_1(0) = (\mu + X(\omega) - v)^2 \). Since \( \frac{\lambda}{2} > X(\omega) + \mu - v \), \( \Phi_1(0) < \Phi_2(\mu + X(\omega) - v) \).

Obviously, the minimum is thus obtained when \( Y = \max \left(0, X(\omega) + \mu - v - \frac{\lambda}{2} \right) \). Lemma A.5 is proved. \( \square \)

**Proof of Proposition 3.2.**

Thanks to lemmas A.4 and A.5, one only has to prove that there exists \( \lambda > 0 \) such that \( Y_\lambda \) defined in lemma A.5 satisfies the condition (ii) of lemma A.4. We then compute its expectation.

\[
\mathcal{E}_\lambda := \mathbb{E} \left[ \left( X + \mu - v - \frac{\lambda}{2} \right) 1_{X \in [v-\mu+\frac{\lambda}{2},+\infty)} \right]
\]

It is obvious then:

\[
\lim_{\lambda \to 0^+} \mathcal{E}_\lambda = \mathbb{E} \left[ (X - v + \mu)^+ \right], \quad \lim_{\lambda \to +\infty} \mathcal{E}_\lambda = 0.
\]

The existence of a solution \( \lambda^* \in \mathbb{R}_+^* \) such that \( \mathcal{E}_\lambda = \Delta \) comes from the assumption on the continuous distribution of \( X \) and thus the continuity of \( \mathcal{E}_\lambda \). Thus we have proved the first part of this Proposition. The second part follows easily from the first part.
References


