OPTIMAL INVESTMENT UNDER PROBABILITY CONSTRAINTS

Carole Bernard[†] **and Steven Vanduffel**[§][1]

[†]*Department of Statistics and Actuarial Science, University of Waterloo, Canada.*
[§]*Faculty of Economic, Political and Social Sciences and Solvay Business School, Vrije Universiteit Brussel, Belgium.*
Email: c3bernar@uwaterloo.ca, steven.vanduffel@vub.ac.be

Abstract

Bernard and Boyle (2010) derive the lowest cost strategy (also called "cost-efficient" strategy) that achieves a given wealth distribution. An optimal strategy for a profit seeking investor with law-invariant preferences is necessarily cost-efficient. In the specific case of a Black-Scholes market the optimal strategy is always path-independent and non-decreasing with the stock price. Assuming now that investors still want to achieve a given distribution at a fixed horizon but have a probability constraint, we propose an explicit construction of the optimal strategy. In the case of the Black-Scholes market, we show that the optimal strategy is not necessarily non-decreasing in the stock price any more.

1. INTRODUCTION

This note extends Bernard and Boyle (2010) by including additional probability constraints. An investor with law-invariant preferences but with some probability constraints has "state-dependent" preferences. We show that the non-decreasing property of the optimal investment for law-invariant preferences does not hold when preferences are state-dependent. Section 2 gives our assumptions, the framework and recalls what cost-efficiency is and its link with optimal investment. Section 3 provides some theoretical results on bounds on copulas under probability constraints and how to use them to solve our optimization problem. We apply theoretical results of Section 3 to some optimal investment problems in Section 4.

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2. COST-EFFICIENCY & OPTIMAL INVESTMENT

In this section we first present the model assumptions and the setting. We then give the general form of the optimal investment problem we want to solve in this paper. In particular we relate the optimal investment choice to the concept of "cost-efficiency" (originally defined by Dybvig (1988a,b)).

2.1. Agent's Preferences

Denote by $U(\cdot)$ the investor's objective function he wants to maximize. We make the following assumptions.

- All investors have a fixed investment horizon $T > 0$ and there is no intermediate consumption.
- Investors prefer "more to less", in other words their respective objective functions preserve first order stochastic dominance relationships (denoted by $\prec_{fsd}$). Hence if $Y_T \prec_{fsd} X_T$ then $U(X_T) \geqslant U(Y_T)$ and $U(\cdot)$ is non-decreasing.
- Investors have "state-independent preferences" or "law-invariant preferences": if Y_T has the same distribution as X_T then $U(Y_T) = U(X_T)$.

Such set of preferences is quite general and consistent with a wide range of decision theories, including the expected utility theory (von Neumann and Morgenstern (1947)), Yaari's dual theory of choice (Yaari (1987)), the cumulative prospect theory (Tversky and Kahneman (1992)) and the rank dependent utility theory (Quiggin (1993)). For example, in the particular case of expected utility the preferences for a final wealth X_T would be calculated as $U(X_T) = \mathbb{E}[u(X_T)]$ where u is the investor's utility function. Instead of maximizing an objective function, one may also minimize any law-invariant risk measure that preserves first stochastic dominance (for example the quantile or a general distorted expectation).

2.2. Financial Market

The financial market contains a (risk-free) bond with price process $\{B_t = B_0 e^{rt},\ t \geqslant 0\}$. Further, there is also a risky asset S with price process $\{S_t,\ t \geqslant 0\}$. We assume trading can be done continuously, the market is frictionless and arbitrage-free, and all investors agree on the pricing kernel used to value derivatives in this market. The initial price $c(X_T)$ of a given contract with payoff X_T maturing at the fixed horizon $T > 0$ is given by

$$c(X_T) = \mathbb{E}[\xi_T X_T]. \tag{1}$$

Here the expectations are taken with respect to the physical probability measure $\mathbb{P}$, and $\{\xi_t,\ t \geqslant 0\}$ is called the state-price process. We will also assume that ξ_t is continuously distributed. In particular it holds that

$$c(1) = \mathbb{E}[\xi_T] = e^{-rT}. \tag{2}$$

It is also well-known that $c(X_T)$ can be presented as the discounted expectation under the risk-neutral measure $\mathbb{Q}$ defined through $\xi_t = e^{-rt}(\frac{d\mathbb{Q}}{d\mathbb{P}})_t$. In the remainder of the paper all expectations are taken under the $\mathbb{P}$ measure. We refer to Bjork (2004) for extensive theory on arbitrage-free pricing.

Note that the above description is rather general and includes the Black-Scholes setting in which case the process $\{\xi_T, t \geqslant 0\}$ is known unambiguously. For the ease of exposition we present all the results in the one-dimensional Black-Scholes market[2]. In this setting there is a bijection between the state-price process ξ_t and the risky asset S_t. Recall that the risky asset price S_t evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{3}$$

where $\{W_t,\ t \geqslant 0\}$ is a standard $\mathbb{P}$-Brownian motion and assume $\mu > r$. The state price process $\{\xi_t,\ t \geqslant 0\}$ exists, is unique and is given by

$$\xi_t = a\left(\frac{S_t}{S_0}\right)^{-\frac{\theta}{\sigma}}, \tag{4}$$

where $a = e^{\frac{\theta}{\sigma}(\mu - \frac{\sigma^2}{2})t - (r + \frac{\theta^2}{2})t}$ and $\theta = \frac{\mu - r}{\sigma}$. Note that ξ_t is decreasing in S_t. Denote by F_ξ the cdf of ξ_T. Let M_T denote the mean of $\log(\xi_T)$, $M_T = -\frac{1}{2}\theta^2 T - rT$. The variance of $\log(\xi_T)$ is equal to $\theta^2 T$. Then,

$$F_\xi(x) = P(\xi_T \leq x) = \Phi\left(\frac{\log(x) - M}{\theta\sqrt{T}}\right). \tag{5}$$

2.3. Cost Efficiency & Investment

The concept of "cost-efficiency" was first introduced by Cox and Leland (1982, 2000) and Dybvig (1988a,b).

Definition 2.1 *A strategy (or a payoff) is cost-efficient if any other strategy that generates the same distribution costs at least as much.*

It is clear that if investors prefer more to less (as per our assumptions in Section 2.1), then in the absence of additional constraints optimal investment strategies will necessarily be cost-efficient. Given the cdf that the investor would like to achieve at a given maturity date T (possibly a retirement date), the optimal strategy then solves the following problem

$$(\mathcal{P}_1) \qquad \begin{array}{l} \min\limits_{X_T} \ \mathbb{E}\left[\xi_T X_T\right] \\ \text{subject to } \ \forall x \in \mathbb{R},\ \ \mathbb{P}(X_T \leqslant x) = F(x) \end{array} \tag{6}$$

The objective is to minimize the cost of a payoff X_T such that X_T has cdf F. Define F^{-1} as follows

$$F^{-1}(y) = \inf\left\{x \mid F(x) \geqslant y\right\}.$$

[2] It would be possible to be more general and include the multidimensional case as studied by Bernard et al. (2011) or the Levy market presented in Vanduffel et al. (2011).

The inverse is left-continuous and non-decreasing. Theorem 2.1 characterizes the optimal investment strategy.

Theorem 2.1 *Let F be a cdf. The solution to $\mathcal{P}_1$ given by* (6) *is equal to*

$$Y_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right), \tag{7}$$

and it is the almost surely unique optimal solution to (6).

This theorem corresponds to the main result of Bernard and Boyle (2010). We will see that it can be obtained as a special case of our approach.

Assume now that the investor is subject to additional constraints that are "state-dependent". The cost-efficient strategy (7) solution to $\mathcal{P}_1$ may not satisfy these constraints and therefore the optimal strategy may be strictly more expensive. We formulate the problem as follows.

$$(\mathcal{P}_2) \qquad \begin{array}{l} \min\limits_{X_T} \ \mathbb{E}\left[\xi_T X_T\right] \\ \text{subject to} \ \left\{ \begin{array}{l} \forall x \in \mathbb{R}, \ \ \mathbb{P}(X_T \leqslant x) = F(x) \\ (\mathcal{C}_i)_{i \in I} \end{array} \right. \end{array} \tag{8}$$

The optimal strategy is distributed with the cdf F but in addition each $\mathcal{C}_i$ denotes an additional constraint and I can be finite or infinite. Each constraint $\mathcal{C}_i$ contains information about the dependency structure between the state-price process and the optimal strategy of the investor given by

$$\mathbb{P}(\xi_T < \ell_i, X_T < x_i) = b_i.$$

In a Black-Scholes market, the state-price process is a function of the risky asset. Then a natural example is a simple probability constraint ensuring that the investment strategy is greater than some guaranteed level when the market itself is very low. The constraint can then write as

$$\mathbb{P}(S_T < \alpha S_0, X_T > b) \leqslant \varepsilon,$$

where $\alpha < 1$, see equation (4).

Adding such constraints is important because investors have state-dependent constraints. For example an investor who invests in a put option, is not interested in cost-efficiency only (because it is decreasing in the underlying stock) but wants positive outcomes when the market goes down.

3. SOLUTIONS TO PROBLEMS $(\mathcal{P}_1)$ AND $(\mathcal{P}_2)$

3.1. Formalization

Problems $\mathcal{P}_1$ and $\mathcal{P}_2$ presented above can be reformulated as "dependence" problems (in other words as problems on copulas). Indeed Problem $\mathcal{P}_1$ is clearly a minimization of $\mathbb{E}[X_T \xi_T]$ where marginals of X_T and ξ_T are known but where no information about the dependency between X_T and ξ_T is given. It can also be interpreted as the minimization of $\mathbb{E}[X_T g(S_T)]$ where marginals of S_T and X_T are known and where $g(y) = a(y/S_0)^{-b}$ for some $b > 0$ because of (4). Problem $\mathcal{P}_2$ is similarly a minimization of $\mathbb{E}[X_T\xi_T]$ or $\mathbb{E}[X_T g(S_T)]$ but with some information on the dependency between X_T and the market S_T.

Let (X,Y) be a couple of random variables. It is well-known that the joint distribution for (X,Y) is fully determined upon knowledge of the marginal distributions F_X and F_Y together with the copula function $C := C_{(X,Y)}$ for (X,Y) (this result is known as Sklar's theorem).

Let us define supermodular functions. Let $\underline{e_i}$ denote the i-th $n$dimensional unit vector, and let $f : \mathbb{R}^n \to \mathbb{R}$ be some function. For $\underline{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we then define $\Delta_i^\varepsilon f(x) = f(\underline{x} + \underline{\varepsilon}\ \underline{e_i}) - f(\underline{x})$ $(\varepsilon_i > 0, 1 \leqslant i \leqslant n)$.

Definition 3.1 (Super modularity) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be supermodular (or 2-increasing) if for all $\underline{x} \in \mathbb{R}^n$, $\delta > 0$, $\varepsilon > 0$ and $1 \leqslant i < j \leqslant n$ it holds that.*

$$\Delta_i^\delta \Delta_j^\varepsilon\ f(\underline{x}) \geqslant 0.$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable then f is supermodular if and only if $\frac{\partial^2}{\partial x_i \partial x_j} f\,(\underline{x}) \geqslant 0$ holds for every $\underline{x} \in \mathbb{R}^n$ and $1 \leqslant i < j \leqslant n$.

See for example Marshall and Olkin (1979), p. 146. A function f is submodular when $-f$ is supermodular.

The problem $\mathcal{P}_2$ given in (8) we want to solve amounts to studying integrals of the form $\mathbb{E}[f(X,Y)]$ where f is submodular or supermodular. Theorem 3.1 below can be found in Tankov (2011) and provides, under suitable assumptions, an expression for the integral $\mathbb{E}[f(X,Y)]$ in terms of the copula C, and the marginal distributions F_X and F_Y.

Theorem 3.1 (Bounds for $\mathbb{E}[f(X,Y)]$) *Assume $f : \mathbb{R}_+^2 \to \mathbb{R}$ is supermodular and left-continuous in each of the arguments. Assume also that*

$$\mathbb{E}\left[|f(X,0)| + |f(0,X)| + |f(Y,0)| + |f(0,Y)| + |f(X,X)| + |f(Y,Y)|\right] < \infty,$$

then $\Pi(C) = \mathbb{E}[f(X,Y)]$ is given by

$$\begin{aligned} \Pi(C) &= -f(0,0) + \mathbb{E}\left[f(X,0)\right] + \mathbb{E}\left[f(0,Y)\right] \\ &\quad + \int_0^\infty \int_0^\infty \mu_f(dx \times dy)(1 - F_X(x) - F_Y(y) + C(F_X(x), F_Y(y)) \end{aligned} \tag{9}$$

where μ_f is the measure on $\mathbb{R}_+^2$ induced by the supermodular function f.

In addition, if the copula C admits pointwise bounds L and U

$$\forall u \in (0,1), \forall v \in (0,1) \quad L(u,v) \leqslant C(u,v) \leqslant U(u,v).$$

Then

$$\Pi(L) \leqslant \Pi(C) \leqslant \Pi(U), \tag{10}$$

where L and U are not necessarily copulas but could be more general functions (such that the double integral in (9) exists).

Proof. The expression (9) is given in Proposition 2 of Tankov (2011). ■

It seems that the expression (9) did not appear yet elsewhere in the literature although it is not the focus of Tankov (2011)[3]. As a first application of Theorem 3.1 let us consider the supermodular function f defined as $f(x, y) = xy$. In this case $\mu_f(dx \times dy) = dx \times dy$. Hence

$$\mathbb{E}[XY] = \int_0^\infty \int_0^\infty \mathbb{P}(X > x, Y > y)dxdy, \tag{11}$$

which is well-known.

Another example of supermodular function is $f(x, y) = -xg(y)$ where $g(y) = a \cdot (y/S_0)^{-b}$. This function appears in the case of a one dimensional Black-Scholes market as the bijection between the risky asset (respectively the market portfolio) and the state price process. In this case, the objective to minimize in problems $\mathcal{P}_1$ and $\mathcal{P}_2$ corresponds to minimizing $\mathbb{E}[f(X_T, S_T)]$. Note that $\frac{\partial^2 f}{\partial x \partial y} \leqslant 0$ which means that it is a submodular function. In that case, $\mu_f(dx \times dy) = g'(y)dxdy$. Hence

$$\mathbb{E}[Xg(Y)] = \int_0^\infty \int_0^\infty \mathbb{P}(X > x, Y > y)g'(y)dxdy. \tag{12}$$

Theorem 3.1 is very useful to actually compute bounds for $\mathbb{E}[f(X, Y)]$ in case one knows the marginal distributions of X and Y, with limited information on the dependence between X and Y. The main idea is to translate the information one has on the dependence to derive bounds on the unknown copula $C_{(X,Y)}$. Using Theorem 3.1 (precisely the inequality (10)), solving problems $\mathcal{P}_1$ and $\mathcal{P}_2$ amounts to finding bounds on copulas. Problem $(\mathcal{P}_1)$ given in (6) and Problem $(\mathcal{P}_2)$ given in (8) can then be formulated as special cases of the following general problem

$$\begin{aligned} &\min_X \ \mathbb{E}\left[f(X, Y)\right] \\ &\text{subject to } \begin{cases} X \sim F, Y \sim G \\ \forall i \in I, \quad \mathbb{P}(Y < \ell_i, X < x_i) = b_i \end{cases} \end{aligned} \tag{13}$$

where I is the set of constraints. Problem $\mathcal{P}_1$ corresponds to $I = \emptyset$. Each additional constraint directly provides information on the dependence between X and Y. In Problem $\mathcal{P}_1$ and $\mathcal{P}_2$, the r.v. Y is the state-price process or a function of S_T, its distribution G is known and depends on the financial market.

The rest of the paper focuses on deriving the bounds A and B such that the unknown copula between X and Y satisfies

$$\forall u, v \in (0, 1), \quad A(u, v) \leqslant C_{(X,Y)}(u, v) \leqslant B(u, v) \tag{14}$$

[3]It generalizes many existing formulas in the literature. For example consider the supermodular function f, $f(x, y) = (x + y - d)_+$. In this case we obtain: $\mu_f(dx \times dy) = \delta_{y=d-x}.dx \times dy$. Hence

$$\begin{aligned} \mathbb{E}[(X + Y - d)_+] &= \mathbb{E}\left[(X - d)_+\right] + \mathbb{E}\left[(Y - d)_+\right] + \int_0^d \mathbb{P}(X > x, Y > d - x)dx \\ &= \mathbb{E}[X] + \mathbb{E}[Y] - d + \int_0^d \mathbb{P}(X \leqslant x, Y \leqslant d - x))dx \end{aligned}$$

which conforms with the expression for $\mathbb{E}[(X + Y - d)_+]$ that was derived in Dhaene and Goovaerts (1996). Their result now appears as a special case of Theorem 3.1.

In general the bounds A and B are not copulas but quasi-copulas. First recall that a two-dimensional copula is any supermodular function $C : [0,1]^2 \to [0,1]$ such that for all u $\in (0,1)$ it holds that $C(0,u) = C(u,0) = 0$ and also that $C(u,1) = C(1,u)$. It is well-known that this definition implies that C is increasing in each argument and also that C is Lipschitz continuous, i.e that $|C(u_1,v_1) - C(u_2,v_2)| \leqslant |u_1 - u_2| + |v_1 - v_2|$ for all (u_1,v_1), $(u_2,v_2) \in [0,1]^2$. These two properties together with the boundary conditions define the weaker concept of quasi-copula:

Definition 3.2 (Quasi-copula) *A two-dimensional quasi-copula is any function* $Q : [0,1]^2 \to [0,1]$ *with the following properties:*

(i) Boundary conditions: for all u $\in (0,1)$ *it holds that* $Q(0,u) = Q(u,0) = 0$ *and also that* $Q(u,1) = Q(1,u)$*;*

(ii) Q *is increasing in each argument and Lipschitz continuous.*

Of course any copula is a quasi-copula but the opposite is not true; for an insightful treatment of copulas we refer to Nelsen (2006). For example a characterization of quasi-copulas is given in Theorem 2.1 of Nelsen et al. (2002).

3.2. Solution to $\mathcal{P}_1$

In Problem $\mathcal{P}_1$, the marginal distributions F_X and F_Y are known but no information is given.

Theorem 3.2 (Classical Fréchet bounds) *Consider a random couple* (X,Y)*, it is well-known that*

$$\forall u,v \in (0,1), \quad \min(u,v) \leqslant C(u,v) \leqslant \max(0, u+v-1)$$

which respectively correspond to the comonotonic and anti-comonotonic copula. Let f *be a supermodular function. Then,*

$$\mathbb{E}\left[f\left(F_X^{-1}(U)\,,\ F_Y^{-1}(1-U)\right)\right] \leqslant \mathbb{E}[f(X,Y)] \ \leqslant \mathbb{E}\left[f\left(F_X^{-1}(U)\,,\ F_Y^{-1}(U)\right)\right].$$

Proof. This result is well-known and the proof is omitted. ■

Solving Problem $\mathcal{P}_1$ is now straightforward and Theorem 2.1 can be seen as a particular case of Theorem 3.2 where $f(x,y) = xy$. For every X_T with cdf F it holds that

$$E[F^{-1}(1 - F_{\xi_T}(\xi_T))] \leqslant \mathbb{E}[\xi_T X_T] \leqslant \mathbb{E}[F^{-1}(F_{\xi_T}(\xi_T))] \tag{15}$$

Note that $(U, 1-U)$ is a legitimate copula so that the bounds are reached.

3.3. Solution to $\mathcal{P}_2$ under probability constraints

We assume that the information on the dependence between X and Y is such that the copula $\mathrm{C}_{(X,Y)}$ is known on a compact subset of the unit square. Bounds were given by Tankov (2011) and we recall here his results

Theorem 3.3 *Let $\mathcal{S}$ be a compact subset of $[0,1]^2$ and consider a quasi-copula Q. Let us define for all u,v $\in [0,1]$*

$$\begin{aligned} U^{\mathcal{S},Q}(u,v) &= \min\left(u, v, \min_{(a,b)\in\mathcal{S}}\left\{Q(a,b)+(u-a)^+ + (v-b)^+\right\}\right), \\ L^{\mathcal{S},Q}(u,v) &= \max\left(0, u+v-1, \max_{(a,b)\in\mathcal{S}}\left\{Q(a,b)-(a-u)^+ - (b-v)^+\right\}\right) \end{aligned} \tag{16}$$

Then for every quasi-copula Q_* so that $Q_*(a,b) = Q(a,b)$ for all $(a,b) \in \mathcal{S}$ it holds that for all $u, v \in [0,1]$

$$L^{\mathcal{S},Q}(u,v) \leqslant Q_*(u,v) \leqslant U^{\mathcal{S},Q}(u,v). \tag{17}$$

Furthermore for all $(a,b) \in \mathcal{S}$ we have that

$$L^{\mathcal{S},Q}(a,b) = U^{\mathcal{S},Q}(a,b) = Q(a,b). \tag{18}$$

Moreover $L^{\mathcal{S},Q}$ and $U^{\mathcal{S},Q}$ are quasi-copulas. Finally, when $\mathcal{S}$ is increasing and Q is a copula, we have that $L^{\mathcal{S},Q}$ is a copula whereas if $\mathcal{S}$ is decreasing, we have that $U^{\mathcal{S},Q}$ is a copula.

Proof. The proof can be found in Tankov (2011). ■

Note that Theorem 3.3 can be applied whenever the values of a copula C are known on a compact subset $\mathcal{S}$ (C just plays the role of Q in this case).

Special case where $\mathcal{S}$ =$\{a,b\}$.
Let C_* a copula such that $C_*(a,b) = \vartheta$ with ϑ such that $\max(a+b-1,0) \leqslant \vartheta \leqslant \min(a,b)$ holds. Then for all $u, v \in [0,1]$ the upper and lower bounds are now given by

$$\begin{aligned} U^{a,b,\vartheta}(u,v) &= \min\left(u, v, \vartheta + (u-a)^+ + (v-b)^+\right), \\ L^{a,b,\vartheta}(u,v) &= \max\left(0, u+v-1, \vartheta - (a-u)^+ - (b-v)^+\right) \end{aligned} \tag{19}$$

respectively. Both are copulas and satisfy $L^{a,b,\vartheta}(a,b) = U^{a,b,\vartheta}(a,b) = C_*(a,b) = \vartheta$. These copulas are called shuffles. In short, a shuffle copula has a support constituted of line segments of slope +1 and -1. More details on shuffles are presented in Section 3.2.3 of Nelsen (2006).

4. EXAMPLES IN BLACK SCHOLES

4.1. Optimization with a unique probability constraint $C(a,b) = \vartheta$

We now describe the simulation of a couple of uniform random variables (U,V) with copula equal to the lower or upper bound found in (19). Draw first a random number u from the uniform (0,1) distribution, then V is fully determined. To obtain a couple (U,V) with the copula $L^{a,b,\vartheta}$, v is calculated as the following function of u

$$\begin{cases} v = 1-u & \text{if } 0 \leqslant u \leqslant a-\vartheta, \\ v = a+b-\vartheta-u & \text{if } a-\vartheta \leqslant u \leqslant a, \\ v = 1+\vartheta-u & \text{if } a \leqslant u \leqslant 1+\vartheta-b, \\ v = 1-u & \text{if } 1+\vartheta-b \leqslant u \leqslant 1. \end{cases} \tag{20}$$

For $U^{a,b,\vartheta}$, it is similar and omitted here. Panel A of Figure 1 gives the support of the shuffle copula $L^{a,b,\vartheta}$.

We now apply this to the construction of the "optimal" solution to $\mathcal{P}_2$ when the probability constraint is given by

$$\mathbb{P}(S_T < \alpha S_0, X_T > b) = \varepsilon \tag{21}$$

where $\alpha > 0$. This probability constraint ensures that the realized payoff is greater than some guaranteed level b when the market itself is low (case when $\alpha < 1$).

In the Black-Scholes model, $S_T = g(\xi_T)$ where g is non-increasing therefore

$$\begin{aligned}\mathbb{P}(S_T < \alpha S_0, X_T > b) &= \mathbb{P}(\xi_T > \ell, X_T > b)\\ &= \mathbb{P}(G(\xi_T) > G(\ell); F(X_T) > F(b))\\ &= 1 - G(\ell) - F(b) + C(G(\ell), F(b))\end{aligned}$$

where $\ell = g(S_0)$ and where C is the copula of (ξ_T, X_T). We are solving a special case of the problem $(\mathcal{P}_2)$ given in (8),

$$\min_{X_T} \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to } \begin{cases} X_T \sim F \\ \ln(S_T) \sim \mathcal{N}\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right) \\ \mathbb{P}(S_T < \alpha S_0, X > b) = \varepsilon \end{cases}$$

This can be rewritten in terms of the state-price process. Note then that $\mathbb{P}(\xi_T \leqslant \ell, X_T \leqslant b) = \varepsilon - 1 + F_{\xi_T}(\ell) + F(b)$. Therefore the problem can be restated as

$$\min_{X_T} \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to } \begin{cases} X_T \sim F \\ \ln(\xi_T) \sim \mathcal{N}\left(M_T, V_T\right) \\ C(F_{\xi_T}(L), F(y_0)) = \vartheta \end{cases}$$

where $\vartheta = \varepsilon - 1 + F_{\xi_T}(\ell) + F(b)$ and C is the copula between ξ_T and X_T. We will use Theorem 3.1 where the copula C that appears in the formula is replaced by the copula L of the lower bound. We construct explicitly the optimal strategy by simulating $U = F_{\xi_T}(\xi_T)$ and constructing V following (20) to simulate a couple (U, V) of uniform (0,1) such that the copula is $L^{F_{\xi_T}(\ell), F(b), \vartheta}$. V is a function of U, let h be such that $V = h(U)$. Then the optimal solution to $\mathcal{P}_2$ with the probability constraint (21) given explicitly by

$$F^{-1}(h(F_\xi(\xi_T))).$$

4.2. Example when F is the cdf of a put option and there is one constraint.

Consider a put option with strike K and maturity T, its payoff is $X_T = (K - S_T)^+$. The cost efficient strategy was found in Bernard and Boyle (2010). We first recall their result and study the effect of adding the probability constraint. Let F be the cdf of the payoff of the put option.

Bernard and Boyle (2010) show that the put option is the (a.s.) unique payoff that has the highest possible cost with cdf F. This cdf, F, is

$$F(x) = P(X_T \leqslant x) = \begin{cases} 1 & \text{if } x \geqslant K \\ P(S_T > K - x) = \Phi\left(\frac{\left(\mu - \frac{\sigma^2}{2}\right)T - \log\left(\frac{K-x}{S_0}\right)}{\sigma\sqrt{T}}\right) & \text{if } 0 \leqslant x < K \\ 0 & \text{if } x < 0 \end{cases}$$

It is straightforward to invert it. Define $\nu = \Phi\left(\frac{\left(\mu-\frac{\sigma^2}{2}\right)T - \log\left(\frac{K}{S_0}\right)}{\sigma\sqrt{T}}\right)$ and consider $y \in (0,1)$,

$$F^{-1}(y) = \left(K - S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}\Phi^{-1}(y)}\right)^+$$

Note that $F^{-1}(1) = K$ and $F^{-1}(0)$ is not well defined. The cost-efficient payoff that gives the same distribution as a put option is

$$Y_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right) = \left(K - S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}\left(\frac{M - \log(\xi_T)}{\theta\sqrt{T}}\right)}\right)^+ = \frac{K}{S_T}\left(S_T - \frac{c}{K}\right)^+,$$

where F_ξ is given by (5) (see Theorem 2.1) and where $c = S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}$. $Y_T^\star$ is the optimal solution to $(\mathcal{P}_1)$ (cheapest strategy with cdf F). We now want the cheapest strategy X_T with cdf F and

$$\mathbb{P}\left(\, X_T > b\, ;\ S_T < 0.95 S_0 \,\right) = \varepsilon$$

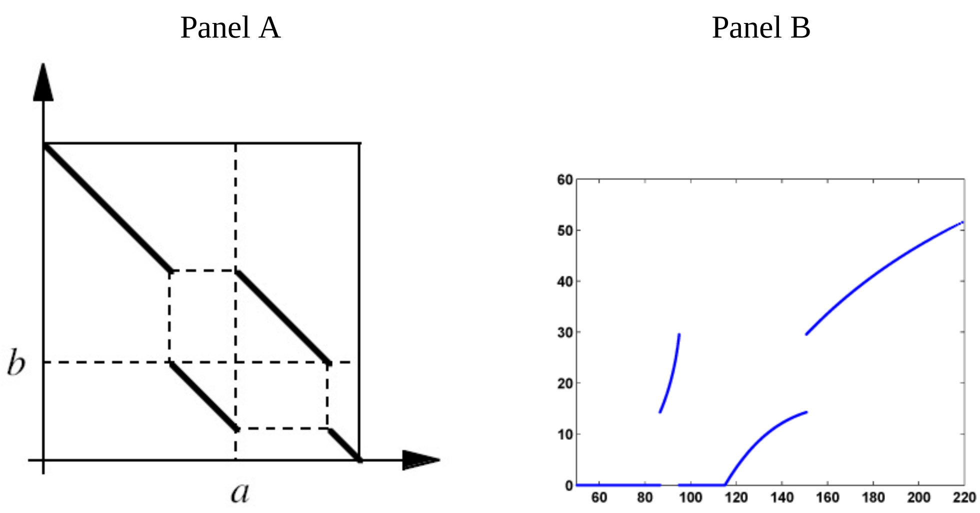

Figure 1: Panel A corresponds to the support of the copula $L^{a,b,\vartheta}$ given by (19). This is an extract from Fig. 3.10 in Nelsen (2006). Panel B displays the cheapest strategy as a function of S_T under the probability constraint under study. Assumptions for Panel B are: $S_0 = 100$, $K = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$, $r = 0.03$, $b = K/7$ and $\varepsilon = .15$.

Panel B in Figure 1 illustrates the optimum.

4.3. Example when F is the cdf of a put option and there is an infinite number of constraints

With several probability constraints, we can solve $(\mathcal{P}_2)$ using the general result in Theorem 3.3. Assume that for all $(a, b) \in I$, the copula between ξ_T and X_T is comonotonic and therefore the copula between X_T and S_T is anti-comonotonic.

$$C(a, b) = \min(a, b)$$

where I is the segment with extremities $(0.7, 0.7)$ and $(1, 1)$. The constraint on the copula applies for $S_T \leq 92.8$ and $X_T \geq 7.21 = F^{-1}(0.7)$. We are looking for the cheapest strategy X_T with cdf F and X_T is anti-comonotonic with the stock market when the stock price is low.

The following figure gives the support of the copula L and the optimal strategy.

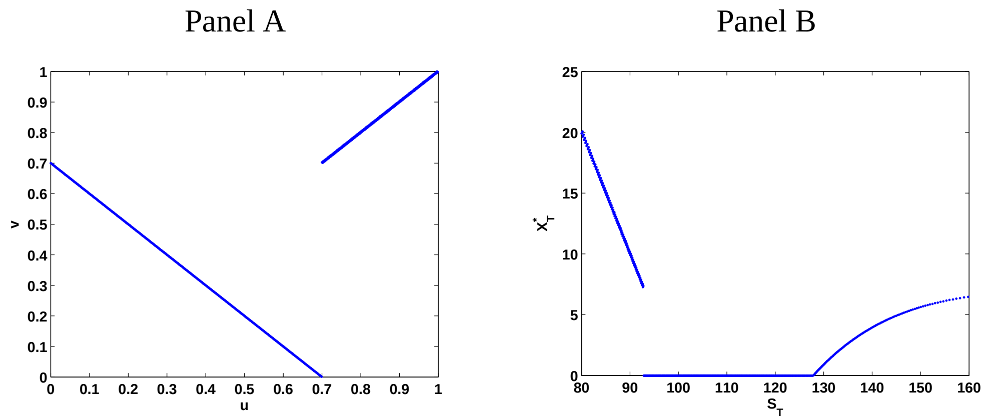

Figure 2: Panel A: Support of the lower bound of the copula between S_T and X_T. Panel B: Optimal Strategy under State-Dependent Constraint. Assumptions: $S_0 = 100$, $K = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$, $r = 0.03$.

Note that Panels B of Figure 1 and Figure 2 both display an optimal strategy under probability constraints that is not non-decreasing with respect to the underlying S_T.

5. CONCLUSIONS

This paper presents optimal investment strategies in the presence of state-dependent constraints. Similarly as Bernard and Boyle (2010) the assumption is that one knows the cdf of terminal wealth and one wants to reach this objective cdf at the cheapest possible cost given some probability constraints. Investors with law-invariant preferences will solely invest in strategies that are non-decreasing in the underlying risky asset. In the presence of probability constraints, non-decreasing strategies in the risky asset are not necessarily optimal.

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