Insurance Market Effects
of Risk Management Metrics

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Abstract

We extend the classical analysis on optimal insurance design to the case when the insurer implements regulatory requirements (Value-at-Risk). Presumably, regulators impose some risk management requirement such as VaR to reduce the insurers' insolvency risk, as well as to improve the insurance market stability. We show that VaR requirements may better protect the insured and improve economic efficiency but have stringent negative effects on the insurance market. Our analysis reveals that insured are better protected in the event of greater loss irrespective of the optimal design from either the insured or the insurer perspective. However, in the presence of the VaR requirement on the insurer, the insurer's insolvency risk might be increased and there are moral hazard issues in the insurance market because the optimal contract is discontinuous.

Keywords: Optimal Insurance Design, Value-At-Risk.

JEL Classification: D18, G22

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1 Introduction

This paper shows that Value-at-Risk (VaR) regulatory requirements have *controversial* effects on the insurance market. *On one hand*, insured are better protected in the event of a large loss irrespective of the optimal design from either the insured's or the issuer's side, when the insurer implements VaR imposed by regulators. *On the other hand*, since the insurer then covers more, when a large loss occurs, the default risk of the insurer is increased as well as the instability of the market. Moreover, because of the presence of discontinuities in the optimal insurance contract presented in this paper, the optimal risk sharing introduces moral hazard issues in the insurance market.

VaR is not a new concept in risk management. Recommendations on banking laws and regulations issued by the Basel Committee on Banking Supervision, have been implemented in the banking sector. In Basel I and Basel II, the VaR methodology is used to deal with the market risk and the credit risk. Similarly, in Solvency II [1] for the insurance sector, the economic capital could be calculated by an internal VaR model. However, whether VaR risk management[2] requirements really enhance the efficiency and stability of the market still remains elusive. For instance, in the context of financial market, Basak and Shapiro (2001) derives that the presence of VaR risk managers amplifies the stock-market volatility in a downward market and attenuates the volatility in an upward market. The 2007-2008 global financial crises lead to even more serious concerns on the adequacy of Value-at-Risk methodology to deal with credit risk. This paper reveals some negative aspects of the regulatory VaR methodology in the insurance market.

We develop a theoretical framework to investigate the economic consequences for the insured and to the insurance market in the presence of regulatory risk management requirements on insurers. The study is done under the expected utility paradigm and the risk management requirement is interpreted as a VaR constraint. The main contributions of this paper are two-fold.

First, we examine the effects on the insurance market when regulators impose a VaR constraint to the insurer. We characterize the optimal insurance contracts from the *insurer*'s perspective to meet VaR risk management constraints. We show that the insurer's

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[1]Solvency II is a new regulatory capital framework for insurance companies initiated by the European Union and starting to be developed in Northern America. We refer to EFMA's report (2006) for its current stage.

[2]Our regulatory risk management is different from the regulatory constraints considered in Raviv (1979). Raviv's description of the regulatory constraint is based on Joskow (1973) and Peltzman (1976). In those previous literature, regulation is endogenous while current regulation becomes compulsory.

optimal insurance contract is a double-capped indemnity (Proposition 4.1). We also derive the optimal insurance contract from the *insured*'s perspective in the presence of risk management constraints imposed on the insurer. Given the VaR constraint, the optimal contract for insured is a capped deductible plus a deductible (Proposition 5.1). To analyze the market effects of the presence of the VaR constraint, we compare the optimal insurance contract in the presence of regulation constraints with the standard results of Arrow (1971) and Raviv (1979). We show that insured obtain *better* protection in the event of *higher* loss and that this higher protection for large loss is compensated by a relatively lower protection against moderate loss. But, the optimal designs are *discontinuous* and therefore introduce moral hazard issues in the insurance market. Furthermore, the expected loss of the insurer is then higher when a large loss occurs. This leads to higher default risk of the insurer, in contrast with the purpose of reducing the default risk of the insurer by the regulatory requirement.

Second, this paper contributes to the optimal insurance design literature by adding a regulatory risk management constraint. Previous results in Arrow (1971), Raviv (1979), Cummins and Mahul (2004) and Golubin (2006) can be viewed as special cases of our results in this paper. Moreover, the technical issues associated with the *non-convexity feature* of the VaR constraint is tackled with the theory of "non-decreasing rearrangement". In fact, the theory of "non-decreasing rearrangements" enables us to verify one revelation principle in our context: it suffices to consider non-decreasing indemnities only. [3]

The approach of this paper is rather theoretical. An alternative is the recent empirical study by Cummins, Dionne, Gagné and Nouira (2007) which indicates that risk management contributes significantly to enhancing efficiency. Another related strand of literature emphasizes the risk management activities of insurer, viewed as financial intermediation (e.g. Froot, Scharfstein and Stein (1993)). One objective of our paper is to compare the optimal insurance contracts *ex ante* and *ex post* the VaR risk management implementation. We don't address the issue of how insurers implement the VaR risk management system. Assuming the VaR has been appropriately implemented by the insurer, the effects on the final wealth of insured as well as insurer, and on the market efficiency are examined.

The remainder of this paper is organized as follows. The model is described in Section 2: the optimal indemnity design problems are presented for both insurer and insured. Feasibility of the constraints involved in these two problems are discussed in Section 3.

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[3]The non-decreasing rearrangement concept was first introduced to the insurance literature by Carlier and Dana (2003, 2005).

Section 4 is devoted to deriving the optimal insurance contract for the insurer and Section 5 the optimal insurance contract for the insured. Section 6 concludes. All proofs are presented in Appendices.

2 The Model

The setting is the standard framework to derive an optimal insurance contract. We consider a risk averse insurer (endowed with non random initial wealth w) and a representative risk averse individual (endowed with non random initial wealth w_0). An insurance policy, $\{I(x), P\}$, provides the reimbursement $I(x)$ when the loss x occurs and P is the premium paid initially to the insurer. We assume that the loss x has a *continuous distribution* but could have a mass point at 0. The randomness of the market is represented by a probability space $(\Omega, \Pr\{\})$. The expectation under $\Pr\{\}$ is denoted by $\mathbb{E}[.]$.

Denote by $c(.)$ the cost faced by the insurer in addition to paying the reimbursement. For the ease of exposition, $c(I) = \eta I, \eta > 0$ [4]. The risk preference of the insurer is represented by a concave utility function $V(x)$ defined over $(0, +\infty)$ and satisfying Inada's conditions, that is $V'(0) := \lim_{x \to 0} V'(x) = +\infty, \quad V'(+\infty) := \lim_{x \to +\infty} V'(x) = 0$. The utility function of the representative insured is denoted by $U(x)$, which is strictly concave and also satisfies Inada's conditions. The insurer's final wealth is given by

$$W = w + P - I - c(I). \tag{1}$$

Assume that $w - v$ is the insolvent trigger level such that whenever $W < w - v$ the insurer is insolvent. Since $W < w - v$ is equivalent to $I(x) > \frac{v+P}{1+\eta}$, then to avoid the insolvency risk of the insurer, the indemnity $I(x)$ must satisfy $I(x) \leqslant \frac{v+P}{1+\eta}$, which means $W \geqslant w - v$. Hence, $I(x) \leqslant \frac{v+P}{1+\eta}$ can be interpreted as a "*solvency condition*". The optimal insurance design problem from insured's perspective, under the solvency condition, is solved by Cummins and Mahul (2004). We extend the solvency condition to the case when it is satisfied with a confidence level. This amounts to set a value-at-risk (VaR) limit on the insurer's risk management imposed for instance by a regulator.

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[4]The method could easily be applied to more general function forms of the cost function where the function $C(\cdot)$ defined by $C(t) = t + c(t)$ is non-decreasing, differentiable over $[0, N]$, and C is invertible.

Precisely, assume v is the VaR limit of the loss of time horizon T with confidence level α. Then, the final wealth of the insurer satisfies:

$$\Pr\{W < w - v\} \leqslant \alpha \tag{2}$$

where both v and α are introduced by regulators. We now present the optimal insurance design problems for both insurer and insured in the presence of the VaR constraint (2).

2.1 From the insurer's perspective

The optimal design problem for the **insurer** in the above framework is as follows:

Problem 2.1. *Find an indemnity $I(x)$ such that*

$$\max_I \{\mathbb{E}\left[V(W)\right]\} :$$

$$\begin{cases} 0 \leqslant I(x) \leqslant x, \\ \mathbb{E}[I(x)] = \Delta, \\ \Pr\{W < w - v\} \leqslant \alpha, \\ I(x) \text{ is non} - \text{decreasing}. \end{cases}$$

The first constraint is standard (see Arrow (1971), Raviv (1979)). Given a premium principle based on the actuarial value of the indemnity, the second constraint can be interpreted as a "*premium constraint*". Indeed, similarly to Raviv (1979), we assume $P = \phi(\mathbb{E}[I(x)])$ where $\phi(x)$ is a general strictly increasing function and $\phi(x) \geqslant x$. Thus, $P = \phi(\Delta)$. It is worth to point out that our paper focuses on the optimal design and does not address the optimal premium (level). The determination of the optimal premium is often solved, by fixing a premium first and next by finding an optimal premium to solve a standard maximum problem in calculus (Raviv (1979); Schlesinger (1981); Meyer and Ormiston (1999)). The third constraint is the VaR constraint (2).

The last constraint prevents downward misrepresentation of the damage by the insured. This constraint is first imposed by Huberman, Mayers and Smith (1983), and it is used to resolve the ex post moral hazard issue. Similarly to Huberman et al. (1983), the non-decreasing assumption is imposed rather than derived from moral hazard implications or the presence of audit costs (e.g. Picard (2000)). Furthermore, this constraint is closely related to the revelation principle (Harris and Townsend (1981), Myerson (1979)): the search for an optimal indemnity schedule can be confined to the schedules under which the insured

has no incentive to misrepresent the damage, that are non-decreasing indemnities only. This revelation principle in our framework is justified by the theory of "non-decreasing rearrangements" (See Carlier and Dana (2003, 2005) and Appendix B below). In other words, the optimal non-decreasing indemnity of Problem 2.1 in which the last constraint is removed is the same as the optimal non-decreasing indemnity of Problem 2.1. This "non-decreasing assumption" on the indemnity plays a key role to tackle the non-convex VaR constraint (2).

2.2   From the insured's perspective

To understand the effects on the insurance market of regulators, we also consider the optimal contract design from the **insured**'s perspective. The insurance contract is written on the aggregate loss x and we investigate the perspective of the representative insured. Under the VaR risk management constraint (2), the optimal contract design for the representative insured is the following:

Problem 2.2. *Find an indemnity $I(x)$ such that*

$$\max_I \left\{ \mathbb{E}\left[U(w_0 - P - x + I(x))\right] \right\} :$$

$$\begin{cases} 0 \leqslant I(x) \leqslant x, \\ \mathbb{E}[I(x)] = \Delta, \\ \Pr\{W < w - v\} \leqslant \alpha, \\ I(x) \text{ is non} - \text{decreasing.} \end{cases}$$

Both Problem 2.1 and 2.2 are subject to the same constraints on $I(x)$. As will be shown in Appendix C, a revelation principle for Problem 2.2 also holds. That is, the non-decreasing optimal indemnity of Problem 2.2 in which the last constraint is removed is the same as the optimal indemnity of Problem 2.2. We impose the "non-decreasing assumption" of the indemnity for the same reason as explained in Problem 2.1.

Previous Literature

Problem 2.2 derives the optimal contract for insured under an exogenous Value-at-Risk constraint imposed by regulators on the insurer. A similar problem has recently been solved by Zhou and Wu (2008). The latter study considers a regulatory constraint on the expected tail risk instead of the probability of the tail risk. Note that a constraint on the expected shortfall is a convex-style constraint and standard techniques can be employed.

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We will compare the optimal contract solving Problem 2.2 with the contract in Zhou and Wu (2008) in Section 5 below.

In another recent paper, Huang (2006) considers the optimal contract for the insured under a constraint as follows:

$$\Pr\{W^i \geqslant \mathbb{E}[W^i] - v\} \geqslant 1 - \alpha. \tag{3}$$

This constraint can be interpreted as a VaR constraint for the insured [5]. While the problem in Huang (2006) is interesting when the insured implements Value-at-Risk, we focus on the effect on the market when the insurer implements VaR. Hence we consider the policyholder's optimal contract under the insurer's VaR constraint. This difference also distinguishes our approach from other previous literature such as Wang et al (2005), Bernard and Tian (2009) in the context of the reinsurance market.

2.3 Benchmark Contracts

In the absence of the VaR constraint (2), that is $\alpha = 1$, Problems 2.1 and 2.2 have been solved in previous literature. These standard results are summarized in the following proposition 2.1 and will be used in our subsequent analysis.

Recall that a capped indemnity is a full insurance up to a capped level, written as $I^c(x) := \min\{x, c\}$, and a deductible indemnity has the indemnification function $I_d(x) := \max\{x - d, 0\}$. For ease of exposition[6], we suppose N, the largest possible loss amount of x, satisfies:

$$0 \leqslant x \leqslant N \leqslant \min\left\{\frac{P + w}{1 + \eta}, w_0 - P\right\}, a.s. \tag{4}$$

which implies that the insurer's wealth and the insured's wealth are both non-negative.

Proposition 2.1. *Let A be a measurable subset of Ω with positive measure $\Pr\{A\}$, and P be a fixed positive number. Assume $0 < \Delta \leqslant \mathbb{E}[x\mathbb{1}_A]$.*

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[5]Note that $\Pr\{W^i \geqslant \mathbb{E}[W^i] - v\} = \Pr\{W^i \geqslant w_0 - v_0\}$, where $v_0 = v + P + \mathbb{E}[x] - \Delta$.

[6]It is possible to accept negative wealth by taking some more general utility functions defined on $(a, +\infty)$ for a negative number a.

1. *There exists a positive number $c > 0$ such that the capped indemnity $I^c(x)\mathbb{1}_A$ solves:*

$$\max_I \left\{ \mathbb{E}\left[V(w + P - (1+\eta)I(x))\mathbb{1}_A \right] \right\} \quad s.t. \quad \begin{cases} 0 \leqslant I(x)\mathbb{1}_A \leqslant x\mathbb{1}_A, \\ \mathbb{E}[I(x)\mathbb{1}_A] \geqslant \Delta. \end{cases} \tag{5}$$

where the cap c is determined by $\mathbb{E}[I^c(x)\mathbb{1}_A] = \Delta$. Moreover $I^c(x)\mathbb{1}_A$ is the unique (almost surely) optimal indemnity solving (5).

2. *There exists a positive number $d > 0$ such that the deductible $I_d(x)\mathbb{1}_A$ solves:*

$$\max_I \left\{ \mathbb{E}\left[U(w_0 - P - x + I(x))\mathbb{1}_A \right] \right\} \quad s.t. \quad \begin{cases} 0 \leqslant I(x)\mathbb{1}_A \leqslant x\mathbb{1}_A, \\ \mathbb{E}[I(x)\mathbb{1}_A] \leqslant \Delta. \end{cases} \tag{6}$$

where the deductible d is determined by $\mathbb{E}[I_d(x)\mathbb{1}_A] = \Delta$. Moreover $I_d(x)\mathbb{1}_A$ is the unique (almost surely) optimal indemnity solving (6).

In the case of $\Pr\{A\} = 1$, the first part is proved by Raviv (1979) and the second part is Arrow (1971)'s deductible optimal contract. In the first part as for the optimal design from the insurer's perspective, the second constraint in (5) means that the premium is greater than or equal to a fixed amount. This makes economic sense because the insurer requires a minimum premium to cover costs. On the other hand, the insured is willing to pay a maximum upfront premium (see the second constraint in (6)). In our subsequent applications the constraint $\mathbb{E}[I(x)\mathbb{1}_A] = \Delta$ is often considered. This proposition also follows from results by Golubin (2006) for $\Pr\{A\} = 1$. When $0 < \Pr\{A\} < 1$, the proof is similar by restricting the space probability to the states of nature in the set A and could be obtained from authors upon request.

3 Feasible Constraints in Problem 2.1 and 2.2

Before solving Problem 2.1 and 2.2 in the next two sections, we need to discuss the feasibility of the constraints first. The purpose of this section is to clarify the set of feasible constraints among $\{\Delta, v, \alpha\}$. In the remainder of this paper, we fix $a = \frac{P+v}{1+\eta}$ and q is the $(1-\alpha)$ quantile of x, that is $\Pr\{x \leqslant q\} = 1 - \alpha$.

The first auxiliary problem solves for the possible range of the VaR parameters $\{v, \alpha\}$ when the actuarial value Δ is given.

Problem 3.1. *Solve the optimal indemnity $I(x)$ such that*

$$\max_I \Pr\{I(x) \leqslant a\} \quad s.t. \quad \begin{cases} 0 \leqslant I(x) \leqslant x, \\ \mathbb{E}[I(x)] \geqslant \Delta. \end{cases}$$

Problem 3.1 solves the maximum probability of the event that the indemnity $I(x)$ is less than a if a minimal actuarial value Δ is guaranteed. This formulation doesn't depend on the premium principle, but if the premium principle is used, it turns out that Problem 3.1 determines the maximum survival probability of the insurer [7] when a minimum premium is charged upfront. The solution of Problem 3.1 is presented in the next proposition.

Proposition 3.1. *Define a function $\Delta_{min}(\cdot)$ by*

$$\Delta_{min}(t) = \mathbb{E}[x\mathbb{1}_{x \leqslant t}] + t\Pr\{x \geqslant t\}, \quad t > 0, \tag{7}$$

and $\Delta_{min} = \Delta_{min}(a)$.

1. *If $\Delta < \Delta_{min}$, the maximum survival probability in Problem 3.1 is one, and any indemnity $I^c(x)$ with $c \leqslant a$ and $\mathbb{E}[I^c(x)] \geqslant \Delta$ is optimal for Problem 3.1.*

2. *If $\Delta_{min} \leqslant \Delta < \mathbb{E}[x]$, then there exists a positive $\lambda \geqslant a$ such that the non-decreasing coverage:*

$$J_{a,\lambda}(x) := I^a(x) + I_\lambda(x) + (\lambda - a)\mathbb{1}_{x > \lambda} \tag{8}$$

is optimal, where λ is determined by solving the $\mathbb{E}[J_{a,\lambda}(x)] = \Delta$. Moreover, among the non-decreasing optimal indemnities, $J_{a,\lambda}(x)$ is unique almost surely.

Proof. See Appendix A. □

The second auxiliary problem is the dual to Problem 3.1. It solves for the range of the actuarial value Δ when the VaR constraint (2) is imposed.

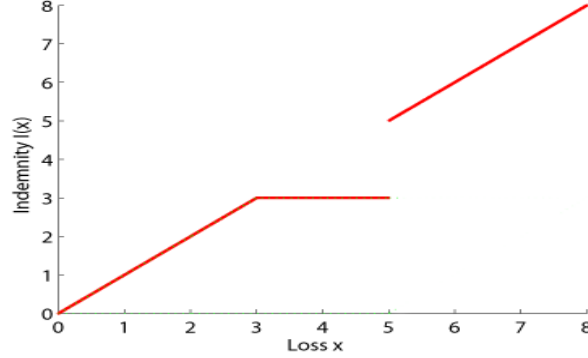**Problem 3.2.** *Solve for the indemnity $I(x)$*

$$\max_I \mathbb{E}[I(x)] \quad s.t. \quad \begin{cases} 0 \leqslant I(x) \leqslant x, \\ \Pr\{I(x) > a\} \leqslant \alpha. \end{cases}$$

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[7]Recall that $I(x) > a$ is equivalent to $W < w - v$.

Figure 1: Optimal Indemnity $J_{a,\lambda}$
This graph displays the optimal insurance contract of maximizing survival
probability with $a = 3$ and $\lambda = 5$. There is a *discontinuity* when $x = \lambda$.

Since the premium is based on the actuarial value, Problem 3.2 is the same as the expected utility for a risk-neutral insured, under the insurer's VaR constraint (2). Problem 3.2 is different from the risk-neutral insured's expected utility problem under the insured's VaR constraint, considered in Wang et al (2005), Bernard and Tian (2009).[8]

Proposition 3.2. *1. If $\Pr\{x > a\} < \alpha$, or equivalently, $q \leqslant a$, then $\Pr\{I(x) > a\} < \alpha$ since $I(x) \leqslant x$. Therefore the full insurance $I(x) = x$ solves Problem 3.2.*

2. If $0 < \alpha \leqslant \Pr\{x > a\}$, or equivalently $a < q$, then the coverage $J_{a,q}(X)$ solves Problem 3.2. Moreover, any non-decreasing optimal indemnity of Problem 3.2 is $J_{a,q}(x)$ a.s.

Proof. See Appendix A. □

Let $\Delta_{max} := \mathbb{E}\left[J_{a,q}(x)\right]$. According to Prop 3.2, Δ_{max} is the maximum possible actuarial value of the indemnity. Hence, to make both Problem 2.1 and Problem 2.2 feasible, we assume:

$$0 < \Delta \leqslant \Delta_{max}, \quad a < q. \tag{9}$$

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[8]Let $I^*(x)$ denotes the optimal solution of the corresponding problem of Problem 3.2 under the VaR constraint of the insured, then $I^*(x)$ has the form $(x - d)^+ \mathbb{1}_{x \leqslant q}$ as shown in Wang et al (2005), Bernard and Tian (2009). Hence the net loss $x - I^*(x)$ has the same shape as $J_{a,\lambda}$. We refer to Bernard and Tian (2009) for more details.

4 Optimal Design for the Insurer

This section presents the optimal design for the risk-averse insurer subject to VaR regulation rules and compares with Raviv's (1979) classical capped indemnity in the absence of the VaR constraint.

For the ease of exposition of the optimal design of the insurer and insured (in the next section), we term a *double-capped indemnity* as $I^{c_1}(x)\mathbb{1}_{x\leqslant q} + I^{c_2}(x)\mathbb{1}_{x>q}$, where $c_1, c_2, q > 0$. A *double-deductible indemnity* is defined as $I_{d_1}(x)\mathbb{1}_{x\leqslant q} + I_{d_2}(x)\mathbb{1}_{x>q}$ for some positive numbers d_1, d_2 and q. A capped deductible indemnity has the form of $\min\{c, I_d(x)\}$ for positive numbers c and d.

To solve Problem 2.1, we first consider the following problem.

Problem 4.1.

$$\max_I \{\mathbb{E}\left[V(W)\right]\} \quad s.t. \quad \begin{cases} 0 \leqslant I(x) \leqslant x, \\ \mathbb{E}[I(x)] = \Delta, \\ \Pr\{W < w - v\} = \alpha. \end{cases}$$

Then, Problem 2.1 is easily solved after characterizing the optimal indemnity of Problem 4.1 for a general probability parameter α. Later we will rationalize this approach because the VaR constraint (2) is not necessarily binding in Problem 2.1.

Proposition 4.1. *For any $0 < \Delta \leqslant \Delta_{max}$, $a < q$, where q is the $(1-\alpha)$ quantile of x,*

1. *If $\Delta \leqslant \Delta_{min}$, then there exists a positive $c \leqslant a$ such that the capped indemnity $I^c(x)$ is the optimal indemnity of Problem 2.1.*

2. *If $\Delta_{min} < \Delta \leqslant \Delta_{max}$, then there exists a positive $c > a$ such that the double-capped indemnity $I^a(x)\mathbb{1}_{x\leqslant q} + I^c(x)\mathbb{1}_{x>q}$ is the optimal indemnity of Problem 4.1. Moreover, the non-decreasing optimal indemnity of Problem 4.1 is unique almost surely. If $\Delta = \Delta_{max}$, this is in fact $J_{a,q}(x)$.*

Proof. The first part of this proposition follows from Proposition 2.1 easily. If $\Delta \leqslant \Delta_{min}$, then there exists a positive $c \leqslant a$ such that $\Delta = \Delta_{min}(c)$. Hence by Proposition 2.1, the capped indemnity $I^c(x)$ is the optimal indemnity subject to the premium constraint that $E[I(x)] = \Delta$. Clearly $I^c(x)$ is also optimal to Problem 2.1 since $\Pr\{W < w - v\} = 0$ and the VaR constraint is redundant. For the second part, see Appendix B. $\square$

The proof of the second part of Proposition 4.1 is quite complicated. We explain briefly the main ideas. The proof is divided into three steps. The first step verifies the revelation principle: it suffices to consider non-decreasing indemnities $I(x)$ only. This step makes use of "non-decreasing rearrangements". In the second step, by considering non-decreasing indemnities only, the optimal indemnity of Problem 4.1 is characterized as a double-capped indemnity $(I(x) = I^{c_1}(x)\mathbb{1}_{x \leqslant q} + I^{c_2}(x)\mathbb{1}_{x > q})$. The last step is the most technical one. We show that, the first cap level c_1 in the optimal indemnity is equal to a. Thus, there is only one unknown parameter, the second cap level c, which is determined by the actuarial value $\mathbb{E}[I(x)] = \Delta$. When $\Delta = \Delta_{max}$, the result is consistent with the Proposition 3.2.

The optimal indemnity of Problem 4.1 can be written as a combination of some simple indemnities. If the second cap $c \leqslant q$, the optimal indemnity can be written as $I^a(x) + (c - a)\mathbb{1}_{x > q}$, a capped indemnity plus an indemnity which pays a constant amount $c - a$ only when the loss amount x is (strictly) greater than a[9]. This indemnity is called a "*digital indemnity*" since it corresponds to a digital option contract in the financial market. If $c > q$, there are three components in the optimal contract. The first one is the capped indemnity $I^a(x)$, the second one is a digital indemnity $(q - a)\mathbb{1}_{x > q}$, while the last one is a capped deductible indemnity with deductible q and cap level $c - q$. In the presence of the digital indemnity, both insurer and insured are willing and *able* to shift the reimbursements from moderate to the large level of loss. If this kind of digital indemnity is absent, there are not enough (Arrow-Debreu) securities to build the optimal policy. In the next section, we derive the same results from the insured's perspective as well.

A remarkable feature in the optimal design is its discontinuous-indemnity ingredient in the event $\{x > q\}$. This discontinuous ingredient is pervasive in the optimal design when a probability constraint is involved, and it appeared also in Bernard and Tian (2009), Gollier (1987), Gajek and Zagrodny (2004) and in Propositions 3.1 and 3.2 of the last section. The presence of a discontinuity introduces moral hazard: if the loss is slightly lower than the threshold, it is optimal for the insured to increase the loss. It thus gives incentives of the insured to manipulate the actual loss.[10]

According to Prop 4.1, the non-decreasing optimal indemnity of Problem 4.1 is unique. This uniqueness result is somewhat surprising. The subtle issue is the *non-convex* style

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[9]In the sense of probability, there is no difference between $x > a$ and $x \geqslant a$.

[10]Implementing discontinuous indemnities is not easy in practice. One way to resolve this moral hazard issue is to impose another constraint that the retention $x - I(x)$ is nondecreasing. Combining the non-decreasing assumption of $I(x)$, we see that $I(x)$ is 1-Lipschitz and hence continuous. We refer to Huberman, Mayes and Smith (1983), Carlier and Dana (2003) for more discussions on non-decreasing retentions.

constraint $\Pr\{W < w - v\} = \alpha$. Hence, there are no available standard results about the existence and the uniqueness when the constraint is non-convex (see Luenberger (1971)).

We now study how the second cap level c depends on α, or equivalently q. To emphasize this dependence we write $c = c(q)$ by viewing q as a variable. Let $g(x)$ denotes the density function of the loss variable. By Proposition 4.1, we obtain, if $c(q) \geqslant q$,

$$\int_0^a xg(x)dx + \int_a^q ag(x)dx + \int_q^{c(q)} xg(x)dx + \int_{c(q)}^N c(q)g(x)dx = \Delta. \tag{10}$$

By differentiating the last equation with respect to the argument q, we obtain:

$$c'(q) \int_{c(q)}^N g(x)dx = (q - a)g(q). \tag{11}$$

In particular, $c'(q) > 0$ and $c(q)$ is increasing with respect to q. Moreover, $\lim_{q \downarrow a} c(q) = c^*$ such that $\mathbb{E}[I^{c^*}(x)] = \Delta$. Since $\Delta > \Delta_{min}$, we have $c^* > a$. Hence, $c(q)$, in the region $\{c(q) \geqslant q\}$, is determined by the equation (11) and the initial boundary condition that $c(a) = c^*$.

To finish the characterization of the function $c(q)$ for all possible parameter q, we consider the region that $c(q) \leqslant q$. By the previous discussion, it might occurred when the curve $c = c(q)$ meets the curve $c = q$ at q^* in the (q, c)-space. If $c(q) \leqslant q$, then by Proposition 4.1,

$$\int_0^a xg(x)dx + \int_a^q ag(x)dx + \int_q^N c(q)g(x)dx = \Delta. \tag{12}$$

By differentiating with respect to q, we obtain

$$c'(q) \int_q^N g(x)dx = (c(q) - a)g(q). \tag{13}$$

In particular $c(q)$ is strictly increasing. Hence $c(q)$, over the region $c(q) \leqslant q$, is determined the equation (13) and a suitable initial boundary condition.

We are now able to solve Problem 2.1.

Because of the uniqueness result in Proposition 4.1, we use $\widetilde{I}^\alpha$ to denote the unique (almost surely) non-decreasing indemnity in Proposition 4.1 for a corresponding parameter α. The next proposition presents the optimal indemnity of Problem 2.1. To highlight the dependence on the probability level α, we employ notation $\Delta_{max}(\alpha)$ instead of Δ_{max}.

Therefore $\Delta_{max}(\alpha) = \mathbb{E}[J_{a,q_\alpha}(x)]$ where q_α is the $(1 - \alpha)$ quantile of x. As a function of the argument α, $\Delta_{max}(\alpha)$ is non-decreasing.

Proposition 4.2. *Assume $\Delta_{min} < \Delta < \Delta_{max}$. Let α_0 satisfy $\Delta_{max}(\alpha_0) = \Delta$. For any $\alpha_1 \in [\alpha_0, \alpha]$, there exists a non-decreasing indemnity $\widetilde{I}^{\alpha_1}(x)$ solving Problem 4.1 where α is replaced by α_1. Let*

$$\hat{\alpha} = \mathrm{argmax}_{\alpha_0 \leqslant \alpha_1 \leqslant \alpha} \ \mathbb{E}\left[V\left(w + P - (1 + \eta)\widetilde{I}^{\alpha_1}(x)\right)\right].$$

Then $\widetilde{I}^{\hat{\alpha}}$ is an optimal solution of Problem 2.1. Moreover, any non-decreasing optimal solution of Problem 2.1 must be a double-capped indemnity.

This proposition follows from Prop 4.1 easily. By continuity arguments and monotonicity of the function $\Delta_{max}(\beta)$ with respect to the argument β, there exists $\alpha_0 \in [0, \alpha]$, such that $\Delta_{max}(\alpha_0) = \Delta$. The existence of $\hat{\alpha}$ is also evident using a continuity argument.

Note that α is the confidence level defined in the VaR constraint (2) and α_0 is determined by the actuarial value Δ. The VaR constraint is binding if and only if $\hat{\alpha} = \alpha$. In contrast with the convex-style constraint problem, however, the VaR constraint (2) is not necessary binding. To see this point, we write $\mathcal{T}(\alpha_1) = \mathbb{E}\left[V\left(w + P - (1 + \eta)\widetilde{I}^{\alpha_1}(x)\right)\right]$, and consider the first-order derivative with respect to q_{α_1}, the $(1 - \alpha_1)$-quantile of x. By abuse of notation, we make use of α, q instead of α_1 and q_{α_1}. By straightforward computation, we have

$$\frac{\partial \mathcal{T}(\alpha)}{\partial q} = g(q)\{V(w + P - (1 + \eta)a) - V(w + P - (1 + \eta)q)\}$$
$$- (1 + \eta)V'\left(w + P - (1 + \eta)c(q)\right)c'(q)\int_{\min\{c(q), q\}}^{N} g(x)dx. \qquad (14)$$

Since $a < q$, $V(w + \Delta - (1 + \eta)a) > V(w + \Delta - (1 + \eta)q)\}$. However, since $V(\cdot)$ is increasing and using (11) and (13), $V'\left(w + P - (1 + \eta)c(q)\right)c'(q)\int_{\min\{c(q), q\}}^{N} g(x)dx > 0$. Therefore $\mathcal{T}(\cdot)$ is not necessarily monotonic with respect to q by formula (14). Consequently, $\mathcal{T}(\alpha_1)$ is not necessarily monotonic with respect to α_1. It means that VaR constraint (2) is not necessarily binding in Proposition 4.2. This point is remarkable since we have to reduce Problem 2.1 to solving a sequence of Problem 4.1 as outlined above. This point, however, is overlooked in Huang (2006). We present some numerical examples in the next section for this important point.

4.1 Numerical Examples

Figure 2: Function $\mathcal{T}(\alpha)$

In those graphs, $w = 10$, $V(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma = 3$, $\eta = 0.05$, $\alpha = 0.1$. The loss x follows either a truncated normal distribution[11] over $[0,8]$ (see Panel A) or a uniform distribution over $[0,8]$ (see Panel B - Panel D). In Panel A, X is a truncated normal $N(4,6)$ with $P = 4.2$, $v = 2$, $\Delta = 3.8532$, $\Delta_{\min} = 3.75$, $\Delta_{\max}(\alpha) = 3.91$, $a = 5.90$, $q_\alpha = 7.1$ and $\alpha_{min} = 0.058$. In Panel B-Panel D, X is uniformly distributed over $[0,8]$. In Panel B, $P = 4.3$, $v = 2$, $\Delta = 3.83$, $\Delta_{\min} = 3.75$, $\Delta_{\max}(\alpha) = 3.91$, $a = 6$, $q_\alpha = 7.2$ and $\alpha_{min} = 0.044$. In Panel C, $P = 4.1$, $v = 2$, $\Delta = 3.73$, $\Delta_{\min} = 3.70$, $\Delta_{\max}(\alpha) = 3.88$, $a = 5.81$, $q_\alpha = 7.2$ and $\alpha_{min} = 0.014$. In Panel D, $P = 4.4$, $v = 3$, $\Delta = 3.97$, $\Delta_{\min} = 3.94$, $\Delta_{\max}(\alpha) = 4.00$, $a = 7.04$, $q_\alpha = 7.2$ and $\alpha_{min} = 0.034$. Our purpose is to show that the shape of $\mathcal{T}(\alpha)$ is rich: increasing, decreasing, hump or other complicated functions.
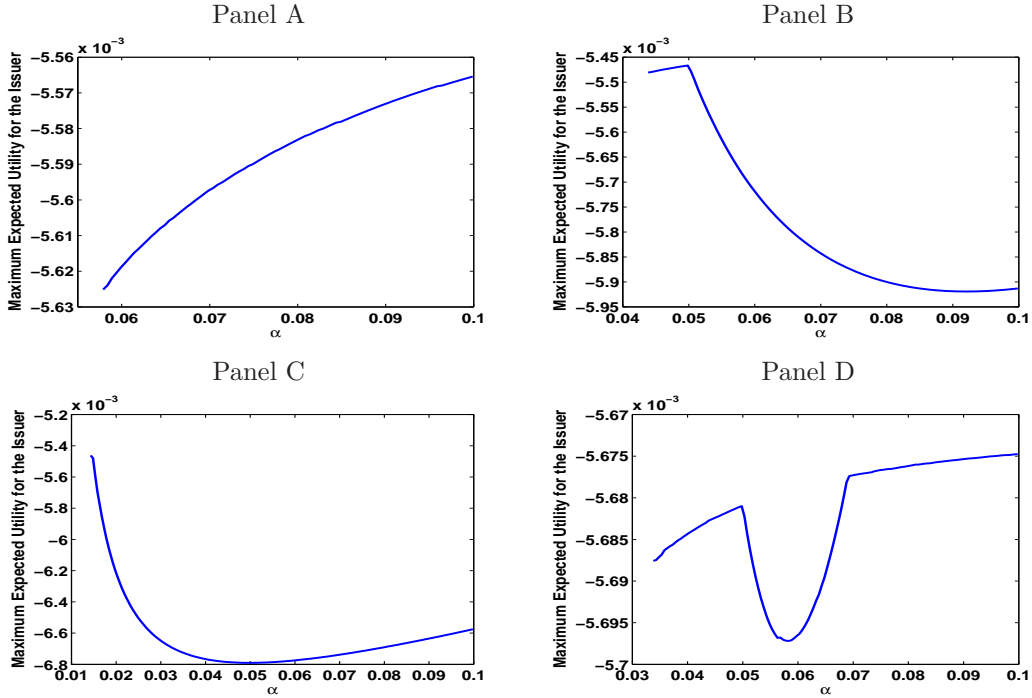
By way of example, we plot several graphs of the function $\mathcal{T}(\alpha_1)$ in Panel A - Panel D of Figure 2. These four graphs clearly show that the function $\mathcal{T}(\alpha_1)$ is not monotonic. Hence the VaR constraint is not necessarily *binding* for the optimal contract of Problem 2.1.

4.2 Analysis

The explicit optimal design derived in the previous section enables us to compare with Raviv's (1979) optimal design without VaR constraint. To illustrate this comparison we

consider two identical insurance companies while one implements the VaR policy that $\Pr\{W < w - v\} \leqslant \alpha$ and another doesn't. We consider the impact to the insurer and the insured separately. For this purpose we assume the same premium P is paid and the insurance contracts are issued based on the optimal design by Proposition 4.1 and Proposition 2.1 respectively. Let a and c be the cap levels of the double-capped indemnity in Proposition 4.1 and r be the cap level in Raviv's indemnity. Clearly, $a < r < c$ because both contracts have the same actuarial value. Both Panel A and Panel B in Figure 3 display the comparisons between the optimal insurance contract with risk management consideration with Raviv's (1979) capped indemnity without risk management consideration. In Panel A the second cap $c > q$, and $c \leqslant q$ in Panel B.

From Figure 3, both companies provide full insurance for insured when the loss $x \leqslant a$. If the moderate loss of x occurs, say $a \leqslant x \leqslant q$, Raviv's optimal contract provides higher protection for the insured. In the event of a high loss of $x > q$, the insurance company which follows the risk management policy actually provides *better* protection to insured. Higher protection for extreme loss looks more attractive for risk-averse insurance buyers.

On the other hand, from the insurer's perspective, Raviv's optimal insurance is not acceptable because it would violate regulatory requirements. To meet the VaR requirement, the insurer has to provide relatively higher indemnity protection when an extreme loss occurs, and consequently, to provide less protection if the loss x occurs in a moderate level $a \leqslant x \leqslant q$. The digital indemnity improves the risk-sharing mechanism, and it enables to shift the indemnity from a moderate level of loss to the coverage of large losses. However, the presence of the discontinuity in the design induces moral hazard: Insured have incentive to inflate losses when it is slightly below q. Moreover, if insurers offer this type of indemnities, then their expected loss in case a loss occurs is greater. It thus increases the default risk of the insurer, which is opposite to the regulatory requirement to reduce the default risk of the insurer. Hence, there is a trade-off between the insured and the insurer and the VaR efffect on the insurance market as a whole is controversial. We will observe the same discoveries for the optimal contract from the insured's perspective in the next section.

Figure 3: Comparison with Raviv's Optimal Design

In both Panel A and Panel B we compare the optimal insurance contract of insurer with Raviv's (1979) insurance contract. The purpose is to show that Raviv's indemnity is *smaller* than insurer's optimal insurance indemnity when the loss $x > q$.

Panel A: $c > q$ Panel B: $c \leqslant q$
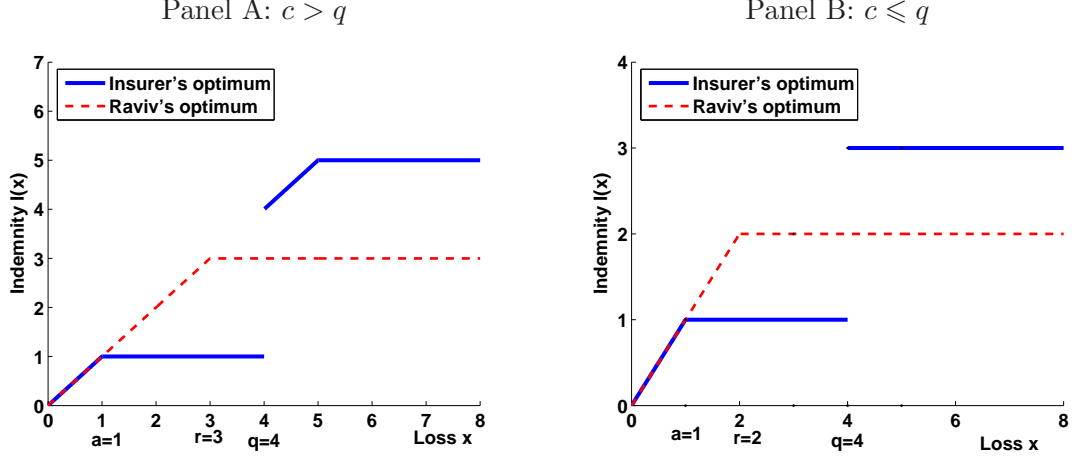
5   Optimal Design for Insured

The previous section presents the optimal insurance design for the insurer. In this section, we discuss the optimal insurance design for insured in the presence of insurer's risk management policies. In the presence of a risk management constraint, Arrow's (1971) deductible policy might not be available because it doesn't meet the VaR requirement for the insurer. Therefore, the risk management constraint for insurer indirectly influences the optimal design for insured.[12]

Similar to Section 4, we reduce Problem 2.2 to the following problem 5.1 where α is a generic probability parameter.

Problem 5.1. *Find the indemnity I such that:*

$$\max_{I} \left\{ \mathbb{E}\left[U(w_0 - P - x + I(x))\right] \right\} \quad s.t. \quad \begin{cases} 0 \leqslant I(x) \leqslant x, \\ \mathbb{E}[I(x)] = \Delta, \\ \Pr\{W < w - v\} = \alpha. \end{cases}$$

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[12]The risk management constraint imposed on the insurer is very different from imposing a risk management constraint to the insured (as in Huang (2006)). In the latter case the insured does not have to take into consideration the insurer's interests. We argue that the VaR constraint to the insurer is important because insurers are regulated while insured are often not (except in the reinsurance market since insured are insurance companies).

The next proposition explicitly characterizes the optimal non-decreasing indemnity for the insured in Problem 5.1, and consequently, solves Problem 2.2.

Proposition 5.1. *For any $0 \leqslant \Delta \leqslant \Delta_{max}$, $a < q$, where q is the $(1 - \alpha)$ quantile of x. Define*

$$\mathcal{M}(q) := \mathbb{E}[(x - (q - a))^+].\tag{15}$$

1. *If $\Delta \leqslant \mathcal{M}(q)$, then there exists a positive $d \geqslant q - a$ such that the deductible indemnity $I_d(x)$ is the optimal indemnity of Problem 2.2. The VaR constraint is redundant.*

2. *If $\mathcal{M}(q) < \Delta \leqslant \Delta_{max}$, then the unique non-decreasing optimal indemnity I^* of Problem 5.1 is written as:*

$$I^*(x) = \min\{a, I_{d^*}(x)\}\mathbb{1}_{x \leqslant q} + I_{d^*}(x)\mathbb{1}_{x>q}$$

where d^* satisfies $\mathbb{E}[I^*(x)] = \Delta$. Moreover, $d^* \leqslant q - a$. If $\Delta = \Delta_{max}(\alpha)$, $I^*(x) = J_{a,q}(x)$.

Proof. The first part of Prop 5.1 follows from Proposition 2.1 [1]: when $\Delta \leqslant \mathcal{M}(q)$, there exists a unique $d \geqslant q - a$ such that $\Delta = \mathbb{E}[(x - d)^+]$. For the remaining proof of this proposition we refer to see Appendix C. □.

The second part of Proposition 5.1 presents the optimal insurance design for the insured under insurer's risk management constraint. A complete proof is fairly lengthy and technical (See Appendix C for details). Thus it is helpful to explain intuitions of the proof. Similar to the proof of Prop 4.1, there are three steps. In the first step, we verify the revelation principle so we confine ourselves to non-decreasing indemnities. The second step is to characterize the optimal design as $\min\{a, I_{d_1}(x)\}\mathbb{1}_{x \leqslant q} + I_{d_2}(x)\mathbb{1}_{x>q}$. The last step is to prove that one particular indemnity is optimal. Namely, $d_1 = d_2$ in the optimal indemnity contract. When $\Delta = \Delta_{max}$, the result is consistent with Proposition 3.2.

Similar to the last section, the sensitivity of d^* with respect to q can be derived explicitly by using Proposition 5.1. However, the VaR constraint (2) is not necessarily binding and hence solution of Problem 2.2 must be derived by a similar procedure of Proposition 4.2.[13]

In a special case $\alpha = 0$, then $q_\alpha = \infty$, and the probability constraint $\Pr\{W < w - v\} = \alpha$ is reduced to $I(x) \leqslant a, a.s.$ Then, the optimal indemnity is $\min\{a, I_{d^*}(x)\}$ which has been

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[13]We omit the details which are available from the authors upon request.

proved by Cummins and Mahul (2004). In the next subsection, we compare the optimal indemnity in Prop 5.1 with Arrow's deductible contract.

5.1 Analysis

Similar to analysis in Section 4.2, we consider two identical companies: one implements the VaR policy while the other one doesn't. A risk-averse insured buys insurance from both insurance companies by paying the same policy premium. One optimal insurance contract is Arrow's deductible contract with deductible level d^{Arrow} while the other optimal insurance contract, by Proposition 5.1, is $I^*(x) = \min\{a, I_{d^*}(x)\}\mathbb{1}_{x \leqslant q} + I_{d^*}(x)\mathbb{1}_{x > q}$ for some q and d^*.

Figure 4: Comparison with Arrow's Optimal Design

This figure displays the optimal insurance contract of insured, under insurer's risk management constraint, with Arrow's (1971) deductible contract. The purpose is to show that Arrow's indemnity is *smaller* when the aggregate loss $x > q$.
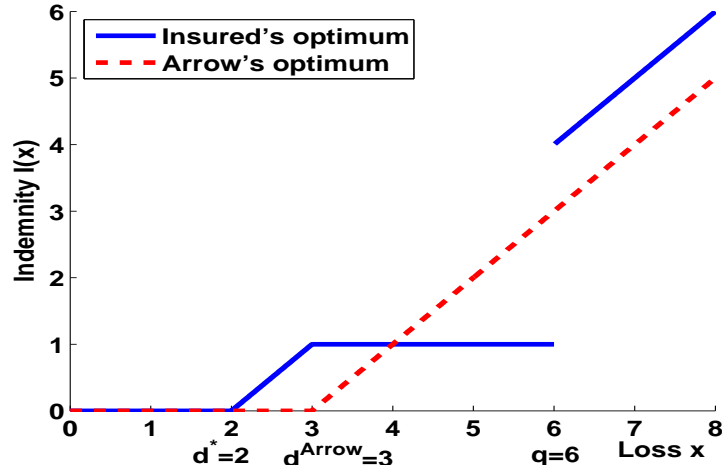
Figure 4 displays both optimal insurance contracts. Since both contracts have the same actuarial value, $d^* < d^{Arrow}$ for any loss distribution of x. Consequently, when $d^* \leqslant x \leqslant d^{Arrow} + a$ or $x > q$, we have $I^*(x) > I_{d^{Arrow}}(x)$. This feature is similar to the optimal design from the perspective of insurer. Moreover, to compensate the higher protection in the event of higher loss, insured are willing to receive reduced indemnity in the event of moderate level of loss. Again, the digital indemnity allows an efficient risk-sharing. The protection amount is shifted from moderate level of loss to higher level of loss. Without digital indemnities, there exists no optimal insurance contract if the insurer follows the VaR constraint. Therefore, the presence of the digital indemnity improves the market efficiency by enhancing risk-sharing opportunities.

Similar to the last section, the insured is better protected in the event of a higher loss because of the VaR requirement on the insurer. In a sense, the VaR requirement has positive effects on the insured since it offers better protection to the insured as desired by insurance regulators. However, it might increase the default risk of the insurer since the amount of loss for the insurer in case of an insurance claim is higher. The smaller d^*, the larger the extra loss $d^{Arrow} - d^*$ over the classical deductible contract. Moral hazard is another issue because of the discontinuity of the optimal contract.

These effects are very specific to Value-at-Risk requirements. As shown in Zhou and Wu (2008), for instance, when the insurer implements the expected shortfall requirements, the insured's optimal indemnity is a capped deductible (clearly, it is continuous). [14] The expected shortfall requirement might be attractive for the insurer since the indemnity is capped. But from the insured's perspective, the default risk is increased. Therefore, different regulatory risk management have shown various effects to the insured and the insurer.

6   Conclusions

We present a theoretical framework to examine the effects of VaR constraints imposed by regulators on insurers, on the optimal form of insurance contracting. When the insurer follows the VaR metric, the optimal insurance designs from both insurer and insured's perspectives are derived explicitly. In the optimal insurance designs, the insured is always better protected in the event of a larger loss. This suggests that the risk management constraint enhances the final wealth of risk-averse insured in the event of a higher loss. Nevertheless, VaR regulation creates moral hazard and increases the default risk of the insurer when large losses occur. According to our results, VaR methodology might be not naturally addressed the risk management of the insurance market.

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[14]Other extensions to general convex risk measure constraints have been investigated by Ludkowski and Young (2008).

Appendices

A   Proof in Section 3: Constraints Feasibility

Proof of Proposition 3.1

We only provide a detailed proof of the second part which requires the following two lemmas.

Lemma A.1. *If Y^* satisfies the three following properties:*
(i) $0 \leqslant Y^* \leqslant x$,
(ii) $\mathbb{E}\left[Y^*\right] = \Delta$,
(iii) there exists a positive $\lambda > 0$ such that for each $\omega \in \Omega$, $Y^*(\omega)$ is a solution of the following optimization problem:

$$\max_{Y \in [0, x(\omega)]} \left\{ \mathbb{1}_{Y \leqslant a} + \lambda Y \right\}$$

then Y^* solves the optimization problem 3.1.

Proof. Given a coverage I which satisfies the constraints of the optimization problem 3.1. Therefore, using (iii), we have,

$$\forall \omega \in \Omega, \quad \mathbb{1}_{Y^*(\omega) \leqslant a} + \lambda Y^*(\omega) \geqslant \mathbb{1}_{I(\omega) \leqslant a} + \lambda I(\omega).$$

Thus, $\mathbb{1}_{Y^* \leqslant a} - \mathbb{1}_{I \leqslant a} \geqslant \lambda \left(I - Y^* \right)$. After taking the expectation, and using condition (ii) one obtains,

$$\Pr\{Y^* \leqslant a\} - \Pr\{I \leqslant a\} \geqslant \lambda \left(\mathbb{E}\left[I\right] - \Delta \right).$$

Therefore, applying the constraints on I, $\mathbb{E}[I(x)] \geqslant \Delta$, $\Pr\{Y^* \leqslant a\} \geqslant \Pr\{I \leqslant a\}$. The proof of this lemma is completed. $\qquad\square$

Lemma A.2. *For each $\lambda > 0$, the indemnity $J_{a, a + \frac{1}{\lambda}}$ (defined by (8) and graphically represented in Figure 1) satisfies the conditions (i) and (iii) of lemma A.1.*

Proof. The property (i) is obviously satisfied. To check the property (iii), we first note that when $x \leqslant a$, the objective function is $1 + \lambda y$ for $0 \leqslant y \leqslant x$. Hence x is the maximum point in this subregion $\{x \leqslant a\}$. When $x > a$, consider two regions of y separately. In the region $y \in [0, a]$, the maximum value of the function $\mathbb{1}_{y \leqslant a} + \lambda y$ is $1 + \lambda a$ at local maximum point a. On the other hand, in the region $y \in (a, x]$, the maximum value of the objective function $\mathbb{1}_{y \leqslant a} + \lambda y$ becomes λx at the local maximum point x. Thus, if $\lambda x > 1 + \lambda a$, or equivalently, $x > a + \frac{1}{\lambda}$, x is the global maximum point; otherwise, a is the global maximum point. Lemma A.2 is proved. $\qquad\square$

Proof of Proposition 3.1:

According to lemmas A.1 and A.2, it suffices to prove the existence of $\lambda^* > 0$ such that $J_{a, a + \frac{1}{\lambda^*}}$ satisfies the condition (ii) of lemma A.1. We compute expectation of $J_{a, a + \frac{1}{\lambda}}$:

$\mathcal{E}_\lambda := \mathbb{E}\left[J_{a,a+\frac{1}{\lambda}}\right]$. We have:

$$\lim_{\lambda \to 0^+} \mathcal{E}_\lambda = \Delta_{\min}(a), \qquad \lim_{\lambda \to +\infty} \mathcal{E}_\lambda = \mathbb{E}[x].$$

The existence of a solution $\lambda^* > 0$ such that $\mathcal{E}_\lambda = \Delta$ comes from the assumption on the continuous distribution of x and thus the continuity of function $\mathcal{E}_\lambda$ of the variable λ.

We now prove that the optimal solution is unique almost surely among the non-decreasing functions of the loss amount x. Let $I(x)$ be another optimal non-decreasing indemnity of problem 3.1 and $I^* = J_{a,a+\frac{1}{\lambda^*}}$. We prove that $I(x) = I^*(x)$ almost surely. In fact, by construction,

$$\forall \omega \in \Omega, \quad \mathbb{1}_{I(x(\omega)) \leqslant a} + \lambda^* I(x(\omega)) \quad \leqslant \quad \mathbb{1}_{I^*(x(\omega)) \leqslant a} + \lambda^* I^*(x(\omega)). \tag{A1}$$

Since $I(.)$ and $I^*(.)$ are both non-decreasing, there exist two positive numbers A and A^* such that $\{I(x) \leqslant a\} = \{x \leqslant A\}$ and $\{I^*(x) \leqslant a\} = \{x \leqslant A^*\}$. Because I and I^* are both optimal, $\Pr\{I(x) \leqslant a\} = \Pr\{I^*(x) \leqslant a\}$. Then $\Pr\{x \leqslant A\} = \Pr\{x \leqslant A^*\}$. As X has a continuous distribution function, one obtains $A = A^*$. Thus, $\mathbb{1}_{I(x) \leqslant a} = \mathbb{1}_{I^*(x) \leqslant a}$ _a.s._ Consequently, using (A1) and $\lambda^* > 0$,

$$I(x) \leqslant I^*(x) \quad a.s.$$

Since $\mathbb{E}[I^* - I] = 0$ and $I^* - I \geqslant 0$ _a.s._, one obtains $I = I^*$ a.s. Prop 3.1 is proved. □

Proof of Proposition 3.2

Its proof is similar to the proof of proposition 3.1. It is straightforward to see that the indemnity $J_{a,a+\lambda}(x(\omega))$ solves the static optimization problem below, for each $\lambda > 0$ and for all ω,

$$\max_{0 \leqslant y \leqslant x(\omega)} \{y - \lambda \mathbb{1}_{y > a}\}. \tag{A2}$$

By using the binding VaR constraint, the optimal λ^* satisfies $a + \lambda^* = q$. Then $J_{a,q}(x)$ is thus the optimal design. The uniqueness of the optimal design is also similar. □

B Proof of Prop 4.1: Optimal design for insurer

We first recall the definition of the "non-decreasing rearrangement" and its key properties (See Carlier and Dana (2005), Proposition 1). Let μ_x be the associated Borel measure on $[0, N]$ of the loss variable x. By assumption on the continuous distribution of x, μ_x is a nonatomic Borel measure.

Definition B.1. _Given a μ_x-Borel function $f : [0, N] \to [0, N]$, there exists a unique (up to μ_x-a.e equivalence) non-decreasing function $\widetilde{f}$ which is equimeasurable to f with respect to μ_x. Moreover, $\widetilde{f}(t) = \inf\{u : v(u) \geqslant t\}$, where_

$$v(u) = \inf\{n \in [0, N] : \mu_x([n, N]) = \mu_x(\{s : f(s) \geqslant u\})\}.$$

$\widetilde{f}$ is called the *non-decreasing rearrangement* of f. Because f and $\widetilde{f}$ are equimeasurable with respect to μ_x, then for all measurable function $g(\cdot)$,

$$\mathbb{E}\left[g\left(f(x)\right)\right] = \mathbb{E}\left[g\left(\widetilde{f}(x)\right)\right]. \tag{B1}$$

Moreover, a variant of Hardy-Littlewood inequality holds. Precisely, if $L(x,t) : [0,N]^2 \to \mathbb{R}$ is $\mathcal{C}^1$ such that for all $t \in [0,N]$, the application $x \to \frac{\partial L}{\partial t}(x,t)$ is increasing, then

$$\mathbb{E}\left[L\left(x,\widetilde{f}(x)\right)\right] \geqslant \mathbb{E}\left[L(x,f(x))\right]. \tag{B2}$$

In the following discussion, given an indemnity form $I(\cdot) : [0,N] \to [0,N]$, there exists a non-decreasing rearrangement of $I(\cdot)$ which is equimeasurable to $I(\cdot)$, with respect to μ_x. If $0 \leqslant I(x) \leqslant x$, then by the proof in Carlier and Dana (2005), Lemma 2, we obtain $0 \leqslant \widetilde{I}(x) \leqslant x$.

Next lemma justifies the revelation principle in the current VaR framework. Moreover, it shows us that the optimal non-decreasing indemnity is a double-capped indemnity.

Lemma B.1. *If there exists an optimal solution I^* to Problem 4.1, then there exists a non-decreasing optimal indemnity $\widetilde{I}^*$ to Problem 4.1. Moreover, any optimal non-decreasing solution of Problem 4.1 is a double-capped indemnity as follows:*

$$\widetilde{I}^*(x) = \begin{cases} I^{c_1}(x) & \text{if } x \leqslant q \\ I^{c_2}(x) & \text{if } x > q \end{cases}$$

where $c_1 \geqslant 0$ and $c_2 \geqslant 0$. Furthermore, $c_1 \leqslant a < c_2$.

Proof. Given an optimal indemnity $I^*(\cdot)$, denote by $\widetilde{I}^*(\cdot)$ the non-decreasing rearrangement of I^*. Clearly, $\widetilde{I}^*$ is also an optimal solution of Problem 4.1. Indeed it satisfies every constraint of Problem 4.1:

$$0 \leqslant \widetilde{I}^*(x) \leqslant x, \quad \mathbb{E}\left[I^*(x)\right] = \mathbb{E}\left[\widetilde{I}^*(x)\right] = \Delta, \quad \Pr\{I^*(x) > a\} = \Pr\{\widetilde{I}^*(x) > a\} = \alpha.$$

Moreover, because of the equi-measurability of $I^*(x)$ and its non-decreasing rearrangement $\widetilde{I}^*(x)$, we have

$$\mathbb{E}\left[V(w + P - (1+\eta)I^*(x))\right] = \mathbb{E}\left[V(w + P - (1+\eta)\widetilde{I}^*(x))\right].$$

Thus $\widetilde{I}^*(x)$ is also optimal. The first part of this lemma has been proved.

We then prove that any non-decreasing optimal solution of Problem 4.1 is a double-capped indemnity. Assume $\widetilde{I}^*$ is a non-decreasing optimal solution of Problem 4.1, then there exist Ω_1 and Ω_2 such that:

- $\Omega_1 = \left\{\widetilde{I}^* \leqslant a\right\}$ and $\Omega_2 = \left\{\widetilde{I}^* > a\right\}$,

- $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega_1 \cup \Omega_2 = \Omega$,

- $\Pr\{\Omega_1\} = 1 - \alpha$ and $\Pr\{\Omega_2\} = \alpha$.

Since $\widetilde{I}^*$ is non-decreasing with respect to x, there exists a constant A such that: $\Omega_1 = \{x \leqslant A\}$ and $\Omega_2 = \{x > A\}$. Recall that q is the $(1 - \alpha)$ quantile of the distribution of x. Because x is continuously distributed, $\Pr\{\Omega_2\} = \alpha$ implies then $q = A$. Thus:

$$\Omega_1 = \{x \leqslant q\} \quad \text{and} \quad \Omega_2 = \{x > q\}.$$

Define $\widetilde{I}_1^*$ and $\widetilde{I}_2^*$ by:
$$\widetilde{I}_1^*(x) = \widetilde{I}^*(x)\mathbb{1}_{x \leqslant q}, \quad \widetilde{I}_2^*(x) = \widetilde{I}^*(x)\mathbb{1}_{x > q},$$

and Δ_1 and Δ_2 such that

$$\mathbb{E}\left[\widetilde{I}_1^*(x)\right] = \Delta_1, \quad \mathbb{E}\left[\widetilde{I}_2^*(x)\right] = \Delta_2, \quad \Delta_1 + \Delta_2 = \Delta. \tag{B3}$$

Let J_1^* and J_2^* be:

$$J_1^*(x) = \begin{cases} J_1(x) & \text{if } x \leqslant q \\ \widetilde{I}_2^*(x) & \text{if } x > q \end{cases} \quad \text{and} \quad J_2^*(x) = \begin{cases} \widetilde{I}_1^*(x) & \text{if } x \leqslant q \\ J_2(x) & \text{if } x > q \end{cases}$$

where J_i, $i = 1, 2$ respectively, solves the following optimization problem:

$$\max_J \{\mathbb{E}\left[V(w + P - (1 + \eta)J(x))\mathbb{1}_{\Omega_i}\right]\} \quad s.t. \quad \begin{cases} 0 \leqslant J(x) \leqslant x, \\ \mathbb{E}[J(x)\mathbb{1}_{\Omega_i}] = \Delta_i. \end{cases}$$

By Proposition 2.1, there exist $c_1, c_2 \geqslant 0$ such that

$$J_i(x) = I^{c_i}(x) \quad \text{a.s. over } \Omega_i \tag{B4}$$

where c_1 and c_2 satisfy $\mathbb{E}[I^{c_i}(x)\mathbb{1}_{\Omega_i}] = \Delta_i$ for $i = 1, 2$.

Claim: $c_1 \leqslant a < c_2$.

Proof of the claim: In fact, $\mathbb{E}[\widetilde{I}^*(x)\mathbb{1}_{x \leqslant q}] = \Delta_1$. Since $\{x \leqslant q\} = \{\widetilde{I}^*(x) \leqslant a\}$, the indemnity $\widetilde{I}_1^*(x)$ stays in the range $[0, \min(x, a)]$ over the region $\{x \leqslant q\}$, then $\Delta_1 \leqslant \mathbb{E}[I^a(x)\mathbb{1}_{x \leqslant q}]$. On the other hand, the function $c \to \mathbb{E}[I^c(x)\mathbb{1}_{x \leqslant q}]$ is non-decreasing, and $\Delta_1 = \mathbb{E}[I^{c_1}\mathbb{1}_{x \leqslant q}]$, then $c_1 \leqslant a$. We now prove that $c_2 > a$. Actually, over $\Omega_2 = \{x > q\}$, the indemnity $\widetilde{I}^*$ is strictly greater than a. Then $\Delta_2 = E[\widetilde{I}^*(X)\mathbb{1}_{x > q}] \geqslant a\alpha > 0$. If $c_2 \geqslant q$ then $c_2 \geqslant a$ (because $q > a$ by assumption). If $c_2 < q$, since $\Delta_2 = E[I^{c_2}\mathbb{1}_{x > q}] = c_2\alpha$, then $c_2\alpha > a\alpha$, hence $c_2 > a$. Thus $c_2 > a$ is proved.

We continue the proof of the lemma B.1.

Since $c_1 \leqslant a$, then $\Pr\{J_1^*(x) > a\} = \alpha$. Moreover, we have $\Pr\{J_2^*(x) > a\} = \alpha$ because $c_2 > a$. For $i = 1, 2$, $\mathbb{E}[\widetilde{I}^*(x)] = \mathbb{E}[J_i^*(x)]$, $0 \leqslant J_i^*(x) \leqslant x$ and $\Pr\{J_i^*(x) > a\} = \alpha$. Therefore $J_i^*(x)$ satisfies the constraints of Problem 4.1. As $\widetilde{I}^*$ is an optimal solution of Problem 4.1,

$$\mathbb{E}\left[V(w + P - (1 + \eta)\widetilde{I}^*(x))\right] \geqslant \mathbb{E}\left[V(w + P - (1 + \eta)J_i^*(x))\right].$$

For $i = 1$, we have

$$\mathbb{E}\left[V(w + P - (1+\eta)\widetilde{I}_1^*(x))\mathbb{1}_{\Omega_1}\right] + \mathbb{E}\left[V(w + P - (1+\eta)\widetilde{I}_2^*(x))\mathbb{1}_{\Omega_2}\right]$$
$$\geqslant \mathbb{E}\left[V(w + P - (1+\eta)J_1(x))\mathbb{1}_{\Omega_1}\right] + \mathbb{E}\left[V(w + P - (1+\eta)\widetilde{I}_2^*(x))\mathbb{1}_{\Omega_2}\right].$$

Thus, $\mathbb{E}\left[V(w + P - (1+\eta)\widetilde{I}_1^*(x))\mathbb{1}_{\Omega_1}\right] \geqslant \mathbb{E}\left[V(w + P - (1+\eta)J_1(x))\mathbb{1}_{\Omega_1}\right]$. Because of the optimality of J_1, and the fact that the solution is unique (a.s.) over Ω_1 (by Prop 2.1),

$$\widetilde{I}_1^* = J_1 = I^{c_1} \quad \text{a.s. over } \Omega_1.$$

The proof is similar when $i = 2$. Lemma B.1 is proved. $\qquad\square$

The next lemma precisely characterizes the optimal indemnity among the set of double-capped indemnities.

Lemma B.2. *Given* $0 \leqslant c_1 < c_1' \leqslant a < c_2' < c_2$, *define* $I = I^{c_1}(x)\mathbb{1}_{x \leqslant q} + I^{c_2}(x)\mathbb{1}_{x > q}$ *and* $I' = I^{c_1'}(x)\mathbb{1}_{x \leqslant q} + I^{c_2'}(x)\mathbb{1}_{x > q}$. *If* $\mathbb{E}[I] = \mathbb{E}[I']$, *then:*

$$\mathbb{E}\left[V\left(w + P - (1+\eta)I\right)\right] \leqslant \mathbb{E}\left[V\left(w + P - (1+\eta)I'\right)\right].$$

Proof. Let $Y_i = w + P - (1+\eta)I^{c_i}$ and $Y_i' = w + P - (1+\eta)I^{c_i'}$. One has $Y_2 \leqslant Y_2' \leqslant Y_1' \leqslant Y_1$. Because of the concavity of $V(.)$, we have

$$(Y_2' - Y_2)\left(V(Y_1) - V(Y_1')\right) \leqslant (Y_1 - Y_1')\left(V(Y_2') - V(Y_2)\right).$$

Over $\{x > q\}$, $Y_1' = w + P - (1+\eta)c_1'$ and $Y_1 = w + P - (1+\eta)c_1$ are constant. Thus, we have

$$\left(V(w + P - (1+\eta)c_1) - V(w + P - (1+\eta)c_1')\right)E\left[(Y_2' - Y_2)\mathbb{1}_{x > q}\right]$$
$$\leqslant (1+\eta)(c_1' - c_1))E\left[(V(Y_2') - V(Y_2))\mathbb{1}_{x > q}\right].$$

By definition, $\mathbb{E}[I] = \mathbb{E}[I']$. Then we have

$$\mathbb{E}\left[(Y_2' - Y_2)\mathbb{1}_{x > q}\right] = \mathbb{E}\left[(Y_1 - Y_1')\mathbb{1}_{x \leqslant q}\right].$$

Then by combining the last two formulae we obtain:

$$\frac{V(w + P - (1+\eta)c_1) - V(w + P - (1+\eta)c_1')}{(1+\eta)(c_1' - c_1)}\mathbb{E}\left[(Y_1 - Y_1')\mathbb{1}_{x \leqslant q}\right] \leqslant \mathbb{E}\left[(V(Y_2') - V(Y_2))\mathbb{1}_{x > q}\right].$$

We now consider the region $\{c_1 \leqslant x \leqslant q\}$. Over this region $\{c_1 \leqslant x \leqslant q\}$, we have

$$w + P - (1+\eta)c_1' \leqslant Y_1' \leqslant Y_1 = w + P - (1+\eta)c_1.$$

By concavity of $V(.)$, for each $\omega \in \Omega$ such that $x(\omega) \in [c_1, q]$, we have

$$(1+\eta)(c_1' - c_1)\left(V(Y_1(\omega)) - V(Y_1'(\omega))\right)$$
$$\leqslant (Y_1(\omega) - Y_1'(\omega))\left(V(w + P - (1+\eta)c_1) - V(w + P - (1+\eta)c_1')\right). \quad \text{(B5)}$$

Moreover, (B5) holds for ω such that $x(\omega) \leqslant c_1$ as well, because $Y_1(\omega) = Y_1'(\omega)$ over $\{x \leqslant c_1\}$. Therefore, (B5) holds for all ω such that $x(\omega) \leqslant q$. Taking expectation over the region $\{x \leqslant q\}$, we obtain a second important inequality:

$$\mathbb{E}\left[(V(Y_1) - V(Y_1'))\,\mathbb{1}_{x \leqslant q}\right] \leqslant \frac{(V(w + P - (1+\eta)c_1) - V(w + P - (1+\eta)c_1'))}{(1+\eta)(c_1' - c_1)} \mathbb{E}\left[(Y_1 - Y_1')\,\mathbb{1}_{x \leqslant q}\right].$$

Therefore, combining the two inequalities in our previous discussion, we obtain:

$$\mathbb{E}\left[(V(Y_1) - V(Y_1'))\,\mathbb{1}_{x \leqslant q}\right] \leqslant \mathbb{E}\left[(V(Y_2') - V(Y_2))\,\mathbb{1}_{x > q}\right].$$

Equivalently,

$$\mathbb{E}\left[V\left(w + P - (1+\eta)I\right)\right] \leqslant \mathbb{E}\left[V\left(w + P - (1+\eta)I'\right)\right].$$

Thus Lemma B.2 is proved. $\qquad\qquad\square$

We are now ready to prove Proposition 4.1. Assume $\Delta_{min} < \Delta < \Delta_{max}$.

Existence of the Optimal Indemnity:

If there exists a non-decreasing solution to Problem 4.1, lemma B.1 implies that the non-decreasing optimal solution must be a double-capped indemnity. Therefore, it suffices to prove that one double-capped indemnity is an optimal solution. Define a function $g(c)$, for $c \geqslant a$ by

$$g(c) = \mathbb{E}[I^a(x)\mathbb{1}_{x \leqslant q}] + \mathbb{E}[I^c(x)\mathbb{1}_{x > q}]. \tag{B6}$$

$g(c)$ is the actuarial value of the double-capped indemnity with the first cap level a, and the second layer cap is c. The function $g(c)$ is increasing and continuous with respect to the argument c. Clearly $g(a) = \Delta_{\min}$, $g(N) = \Delta_{\max}$ and define $\Delta_{med} = g(q)$. Then for any $\Delta \in (\Delta_{min}, \Delta_{max})$, there exists unique $c > a$ such that $g(c) = \Delta$. Moreover, $c \geqslant q$ if and only if $\Delta \geqslant \Delta_{med}$.

We prove that the double-capped indemnity

$$Y_c := I^a(x)\mathbb{1}_{x \leqslant q} + I^c(x)\mathbb{1}_{x > q}$$

is an optimal indemnity of Problem 4.1. For this purpose, let $I = I(x)$ be another indemnity that is non-decreasing and subject to the constraints that $\mathbb{E}[I(x)] = \Delta$, $0 \leqslant I(x) \leqslant x$ and $\Pr\{I(x) > a\} = \alpha$. It suffices to prove that:

$$\mathbb{E}\left[V\left(w + P - (1+\eta)I\right)\right] \leqslant \mathbb{E}\left[V\left(w + P - (1+\eta)Y_c\right)\right].$$

Since I is assumed to be non-decreasing and $\Pr\{I > a\} = \alpha$, then

$$\{I(x) > a\} = \{x > q\}.$$

Let $I_1 = I\mathbb{1}_{x \leqslant q}$ and $I_2 = I\mathbb{1}_{x > q}$, and define by Δ_1 and Δ_2 by

$$\mathbb{E}[I_1] = \Delta_1, \quad \mathbb{E}[I_2] = \Delta_2.$$

Choose c_1 and c_2 such that

$$E[I^{c_1}\mathbb{1}_{x\leqslant q}] = \Delta_1, \quad \mathbb{E}[I^{c_2}\mathbb{1}_{x>q}] = \Delta_2.$$

By using Proposition 2.1, one has:

$$\begin{cases} \mathbb{E}\left[V\left(w + P - (1+\eta)I\right)\mathbb{1}_{x\leqslant q}\right] \leqslant \mathbb{E}\left[V\left(w + P - (1+\eta)I^{c_1}\right)\mathbb{1}_{x\leqslant q}\right] \\ \mathbb{E}\left[V\left(w + P - (1+\eta)I\right)\mathbb{1}_{x>q}\right] \leqslant \mathbb{E}\left[V\left(w + P - (1+\eta)I^{c_2}\right)\mathbb{1}_{x>q}\right] \end{cases}$$

Therefore,

$$\mathbb{E}\left[V\left(w + P - (1+\eta)I\right)\right] \leqslant \mathcal{S} \tag{B7}$$

where $\mathcal{S} := \mathbb{E}\left[V\left(w + P - (1+\eta)I^{c_1}\right)\mathbb{1}_{x\leqslant q}\right] + \mathbb{E}\left[V\left(w + P - (1+\eta)I^{c_2}\right)\mathbb{1}_{x>q}\right]$. Over $\{x \leqslant q\}$, $I(x) \leqslant a$. Then by the same proof of the *Claim* in the proof of lemma B.1, we have $c_1 \leqslant a$. On the other hand, since $\mathbb{E}[I^{c_1}\mathbb{1}_{x\leqslant q} + I^{c_2}\mathbb{1}_{x>q}] = \Delta = \mathbb{E}[Y_c] = \mathbb{E}[I^a\mathbb{1}_{x\leqslant q} + I^c\mathbb{1}_{x>q}]$, and $c_1 \leqslant a$, then $c_2 \leqslant c$. Moreover, $c_1 < a$ implies $c_2 < c$, and $c_1 = a$ implies that $c_2 = c$. By lemma B.2, and note that $c_1 \leqslant a \leqslant c_2 \leqslant c$, we obtain

$$\mathcal{S} \leqslant \mathbb{E}\left[V\left(w + P - (1+\eta)Y_c\right)\right]. \tag{B8}$$

Due to (B7) and (B8), we have proved the existence part.

Uniqueness of the non-decreasing Optimal Indemnity:

Assume there exists another non-decreasing optimal indemnity for Problem 4.1, then by lemma B.1, it must be a double-capped indemnity with first layer cap $c_1 \leqslant a$ and the second layer cap $c_2 > a$. By lemma B.2, since this indemnity is optimal, $c_1 = a$. Therefore, the uniqueness follows from the strictly increasing property of the function $g(c)$. Proposition 4.1 is proved. $\qquad\square$

C   Proof of Prop 5.1: Optimal design for insured

We start with an extension of Prop 2.1 [2] with further constraints on the indemnity $I(x)$.

Lemma C.1. *Let A be a measurable subset of Ω with positive measure $\Pr\{A\}$, and fixed positive numbers P, a and Δ. Assume that $\Delta < \mathbb{E}[x\mathbb{1}_A]$.*

1. *If $\Delta \leqslant \mathbb{E}[\min\{x, a\}\mathbb{1}_A]$, then there exists $d > 0$ such that $\min\{I_d(x), a\}\mathbb{1}_A$ solves the optimal solution of the following problem:*

$$\max_I \left\{\mathbb{E}\left[U(w_0 - P - x + I(x))\mathbb{1}_A\right]\right\} \quad s.t. \quad \begin{cases} 0 \leqslant I(x)\mathbb{1}_A \leqslant x\mathbb{1}_A, \\ I(x)\mathbb{1}_A \leqslant a, \\ \mathbb{E}[I(x)\mathbb{1}_A] = \Delta. \end{cases}$$

where d is determined by $\mathbb{E}[\min\{I_d(x), a\}\mathbb{1}_A] = \Delta$. Moreover $\min\{I_d(x), a\}\mathbb{1}_A$ is the unique optimal indemnity subject to the corresponding constraints (a.s.).

2. *If $\mathbb{E}[a\mathbb{1}_A] \leqslant \Delta \leqslant E[\max\{x,a\}\mathbb{1}_A]$, then there exists $d > 0$ such that $\max\{I_d(x), a\}\mathbb{1}_A$ solves the optimal solution of the following problem:*

$$\max_I \{\mathbb{E}\left[U(w_0 - P - x + I(x))\mathbb{1}_A\right]\} \quad s.t. \quad \begin{cases} 0 \leqslant I(x)\mathbb{1}_A \leqslant x\mathbb{1}_A, \\ I(x)\mathbb{1}_A \geqslant a\mathbb{1}_A, \\ \mathbb{E}[I(x)\mathbb{1}_A] = \Delta. \end{cases}$$

where d is determined by $\mathbb{E}[\max\{I_d(x), a\}\mathbb{1}_A] = \Delta$. Moreover $\max\{I_d(x), a\}\mathbb{1}_A$ is the unique optimal indemnity subject to the corresponding constraints (a.s.).

Proof. When $A = \Omega$, a similar result has been presented in Cummins and Mahul (2004) for an upper limit on coverage under an actuarial value constraint $\mathbb{E}[I(x)] = \Delta$. It can also be proved similarly as Proposition 2.1. Hereafter, we provide the proof when $\Pr\{A\} = 1$. In the general case the proof is the same by adding $\mathbb{1}_A$ throughout the discussion. The proof of the second part is similar and omitted.

Given $\lambda > 0$, consider the following optimization problem:

$$\max_{0 \leqslant y \leqslant x, \ y \leqslant a} \{U(w_0 - P - x + y) - \lambda y\} \tag{C1}$$

for a parameter $\lambda \geqslant U^{-1}(w_0 - P)$. It is easy to verify that $\min\{(x - (w_0 - P - [U']^{-1}(\lambda))^+, a\}$ solves the above optimization problem. Since $\Delta \leqslant \mathbb{E}[\min\{x, a\}]$, there exists $\lambda^* \geqslant U^{-1}(w_0 - P)$ such that

$$\mathbb{E}\left[\min\left\{\left(x - (w_0 - P - [U']^{-1}(\lambda^*)\right)^+, a\right\}\right] = \Delta.$$

Thus a capped deductible $\min\{I_d(x), a\}$ with $d = w_0 - P - [U']^{-1}(\lambda^*)$ solves the optimal insurance design problem. The proof is done when $\Pr\{A\} = 1$. $\qquad\square$

Before proving Proposition 5.1, we present two lemmas which characterize the optimal indemnity among a group of non-decreasing indemnities with the form $\min\{a, I_{d_1}\}\mathbb{1}_{x \leqslant q} + I_{d_2}\mathbb{1}_{x > q}$ or $\min\{a, I_{d_1}\}\mathbb{1}_{x \leqslant q} + \max\{f, I_{d_2}\}\mathbb{1}_{x > q}$, respectively.

Lemma C.2. *Given $\alpha > 0$, q the $(1 - \alpha)$ quantile (which greater than a by assumption), $d_1 \in [0, q]$ and $d_2 \in [0, q - a]$. Denote by I_{d_1, d_2} as the following indemnity:*

$$I_{d_1, d_2} = \min\{a, I_{d_1}\}\mathbb{1}_{x \leqslant q} + I_{d_2}\mathbb{1}_{x > q} \tag{C2}$$

where d_1 and d_2 satisfy $\mathbb{E}[I_{d_1, d_2}] = \Delta \in (\mathcal{M}(q), \Delta_{max}(\alpha))$. There exists a unique $d^* \in [0, q - a]$ such that:
$$\mathbb{E}[I_{d^*, d^*}] = \Delta.$$

Moreover, for all $d_1 \in [0, q]$ and $d_2 \in [0, q - a]$ such that $\mathbb{E}[I_{d_1, d_2}] = \Delta$, we have

$$\mathbb{E}\left[U\left(w_0 - P - x + I_{d_1, d_2}(x)\right)\right] \leqslant \mathbb{E}\left[U\left(w_0 - P - x + I_{d^*, d^*}(x)\right)\right].$$

Proof. Since $\mathcal{M}(q) = \mathbb{E}\left[(X - (q - a))^+\right]$ and using the continuity of the distribution of the loss x and thus of the mapping $d^* \to \mathbb{E}\left[I_{d^*, d^*}\right]$, the existence and the uniqueness of d^* are proved.

To prove the second part, it is worth to explain the idea first. We first interpret d_2 as a function of the variable d_1, say $d_2 = \phi(d_1)$, if $\{d_1, d_2\}$ satisfy $E[I_{d_1, d_2}] = \Delta$. One easily

obtains:

$$d_2 = \phi(d_1) := \frac{\mathbb{E}\left[\min\left((x - d_1)^+, a\right) \mathbb{1}_{x \leqslant q}\right] - \Delta}{\alpha} + \frac{\mathbb{E}\left[x \mathbb{1}_{x > q}\right]}{\alpha}.$$

By construction, for all $d_1 \in [0, q]$, $\mathbb{E}[I_{d_1, \phi(d_1)}] = \Delta$. Denote by g the density of the loss x and differentiate this equation with respect to d_1. Then

$$\phi'(d_1) = -\frac{\int_{d_1}^{\min\{d_1 + a, q\}} g(x)\,dx}{\alpha},$$

which is obviously negative. Hence $d_2 = \phi(d_1)$ is a decreasing function of d_1. Let

$$\Phi(d_1) := \mathbb{E}\left[U\left(w_0 - P - x + I_{d_1, \phi(d_1)}(x)\right)\right].$$

It is straightforward to obtain

$$\Phi'(d_1) = \left(U'\left(w_0 - P - \phi(d_1)\right) - U'\left(w_0 - P - d_1\right)\right) \int_{d_1}^{\min\{d_1 + a, q\}} g(x)dx.$$

Therefore the function $\Phi(.)$ has a unique maximum when $d_1 = \phi(d_1)$, that is when $d_1 = d_2 = d^*$. Indeed, the integral $\int_{d_1}^{\min\{d_1 + a, q\}} g(x)$ is always positive, thus $\Phi'(d_1)$ is positive over $\{d_1 < d^*\}$ and negative over $\{d_1 > d^*\}$ because $U'(\cdot)$ is decreasing. □

Lemma C.3. *Given $\alpha > 0$, q the $(1-\alpha)$-quantile (assumed to be greater than a), $d_1 \in [0, q]$, $d_2 > 0$, $f > a$. Denote by $I_{d_1, d_2, f}$ as the following indemnity:*

$$I_{d_1, d_2, f} := \min\{a, I_{d_1}\}\mathbb{1}_{x \leqslant q} + \max\{f, I_{d_2}\}\mathbb{1}_{x > q} \tag{C3}$$

where $\{d_1, d_2, f\}$ satisfy that $\mathbb{E}[I_{d_1, d_2, f}] = \Delta$. Then:

$$\mathbb{E}\left[U\left(w_0 - P - x + I_{d_1, d_2, f}(x)\right)\right] \leqslant \mathbb{E}\left[U\left(w_0 - P - x + I_{d^*, d^*}(x)\right)\right] \tag{C4}$$

where I_{d^*, d^*} is defined in lemma C.2.

Proof. The proof of this lemma builds on the same ideas as the proof of lemma C.2. We proceed in two steps.

   Step 1: We first assume d_1 is fixed. Then we define (implicitly) d_2 as a function of the floor f by using the equation: $\mathbb{E}\left[I_{d_1, d_2, f}\right] = \Delta$. Write $d_2 = \psi(f)$. Differentiating the equality $\mathbb{E}\left[I_{d_1, \psi(f), f}\right] = \Delta$ with respect to f, we obtain:

$$\psi'(f) \int_{f + \psi(f)}^{N} g(x)dx = \int_{q}^{f + \psi(f)} g(x)dx$$

which proves in particular that $\psi(f)$ is an increasing function of the floor f. Define

$$\Psi(f) := \mathbb{E}\left[U\left(w_0 - P - x + I_{d_1, \psi(f), f}(x)\right)\right].$$

After some computations, we have

$$\Psi'(f) = \int_{q}^{f + \psi(f)} \left(U'\left(w_0 - P - x + f\right) - U'\left(w_0 - P - \psi(f)\right)\right) g(x)dx. \tag{C5}$$

Over the range $x \in (q, f + \psi(f))$, $-\psi(f) < f - x$, thus $\Psi'(f) < 0$. Hence $\Psi(f)$ is decreasing for $f > a$. Therefore, by Lebesgue dominance theorem, for some d_2,

$$\sup_{f>a} \Psi(f) = \lim_n \Psi\left(a + \frac{1}{n}\right) = \mathbb{E}\left[\min\{I_{d_1}, a\}\mathbb{1}_{x \leqslant q}\right] + E[\max\{I_{d_2}, a\}\mathbb{1}_{x>q}]$$

Hence $I_{d_1, d_2, f}$ is dominated by the indemnity of the form $\min\{I_{d_1}, a\}\mathbb{1}_{x \leqslant q} + \max\{I_{d_2}, a\}\mathbb{1}_{x>q}$.

Step 2: In this step we consider the indemnity of the form (by abuse of the notation)

$$I_{d_1, d_2} := \min\{I_{d_1}, a\}\mathbb{1}_{x \leqslant q} + \max\{I_{d_2}, a\}\mathbb{1}_{x>q}$$

with the actuarial value Δ. We prove that the expected utility of the indemnity of the form $\min\{I_{d_1}, a\}\mathbb{1}_{x \leqslant q} + \max\{I_{d_2}, a\}\mathbb{1}_{x>q}$ is dominated by the expected utility of the indemnity with $d_2 = q - a$. If $d_2 \leqslant q - a$, then I_{d_1, d_2} is reduced to the same indemnity I_{d_1, d_2} in Lemma C.2. Therefore, in the subsequent proof, we only consider $d_2 > q - a$.

We now express d_1 as a function of d_2, say $\chi(d_2)$. Write: $\mathbb{E}\left[I_{\chi(d_2), d_2}\right] = \Delta$. Since $\Delta > \mathcal{M}(q)$, then $\chi(d_2) \leqslant q - a$. Moreover,

$$\chi'(d_2) \int_{\chi(d_2)}^{min(q, \chi(d_2)+a)} g(x) = -\int_{d_2+a}^{N} g(x)dx. \tag{C6}$$

Define $\mathcal{X}(d_2) = \mathbb{E}\left[U\left(w_0 - P - x + I_{\chi(d_2), d_2}\right)\right]$, by straightforward computation we have

$$\mathcal{X}'(d_2) = \int_{d_2+a}^{N} \{U'(w_0 - P - \chi(d_2)) - U'(w_0 - P - d_2)\}g(x)dx.$$

Since $d_1 = \chi(d_2) \leqslant d_2$, then $\mathcal{X}'(d_2) < 0$. Therefore $\mathcal{X}(d_2) \leqslant \mathcal{X}(q - a)$ for all $d_2 > q - a$. Then the lemma follows from lemma C.2 easily. $\qquad \square$

Proof of Proposition 5.1: Assume $\mathcal{M}(q) < \Delta < \Delta_{max}$.

Step 1. We verify the revelation principle in this step, that is, if there exists an optimal solution I^* to Problem 5.1, then there exists a non-decreasing optimal indemnity $\widetilde{I}^*$ to Problem 5.1. Indeed, if I is an optimal solution then its non-decreasing rearrangement is also an optimal solution. Constraints are satisfied by $\widetilde{I}$ because of the properties of the non-decreasing rearrangement. Moreover, by Hardy-Littlewood inequality (B2) with $L(x, t) = U(w_0 - P - x + t)$, one has:

$$\mathbb{E}\left[U\left(w_0 - P - x + I(x)\right)\right] \leqslant \mathbb{E}\left[U\left(w_0 - P - x + \widetilde{I}(x)\right)\right]$$

because $x \to U'(w_0 - P - x + t)$ is increasing.

Step 2. Assume that I^* is one non-decreasing optimal indemnity of the Problem 5.1. Because I^* is non-decreasing, and $\Pr\{I^* > a\} = \alpha$, then we have

$$\{I^*(x) > a\} = \{x > q\}, \quad \{I^*(x) \leqslant a\} = \{x \leqslant q\}. \tag{C7}$$

Let $I_1^*(x) = I^*(x)\mathbb{1}_{x \leqslant q}$, $I_2^*(x) = I^*(x)\mathbb{1}_{x>q}$, and $\Delta_i = \mathbb{E}[I_i^*(x)]$ for $i = 1, 2$.

We first deal with $I_1^*(x)$. Clearly $\Delta_1 \leqslant \mathbb{E}[\min\{x, a\}\mathbb{1}_{x \leqslant q}]$. It is also easy to see that $I_1^*(x)$ solves the following optimal problem

$$\max_J \mathbb{E}\left[U\left(w_0 - P - x + J(x)\right)\mathbb{1}_{x \leqslant q}\right]$$

subject to constraints $0 \leqslant J(x)\mathbb{1}_{x \leqslant q} \leqslant x\mathbb{1}_{x \leqslant q}$, $J(x)\mathbb{1}_{x \leqslant q} \leqslant a$ and $\mathbb{E}[J(x)\mathbb{1}_{x \leqslant q}] = \Delta_1$. Then by lemma C.1, there exists $d_1 \geqslant 0$ such that

$$I_1^*(x) = I^*(x)\mathbb{1}_{x \leqslant q} = \min\{a, I_{d_1}(x)\}\mathbb{1}_{x \leqslant q}. \tag{C8}$$

Step 3. We characterize $I_2^*\mathbb{1}_{x > q}$ in this step. Define

$$\mathcal{A}_n := \left\{I^* \geqslant a + \frac{1}{n}\right\}. \tag{C9}$$

Then $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ and $\cup_n \mathcal{A}_n = \{I(x) > a\} = \{x > q\}$. For $n >> 0$ (which means that there exists $p \geqslant 0$ such that for all $n \geqslant p$), by lemma C.1, there exists $d_n \geqslant 0$ such that

$$I^*\mathbb{1}_{\mathcal{A}_n} = I_n := \max\left\{a + \frac{1}{n}, (x - d_n)^+\right\}\mathbb{1}_{\mathcal{A}_n}. \tag{C10}$$

We consider the two cases separately.

Case 1. If for all $n >> 0$, $I_n = (x - d_n)^+\mathbb{1}_{\mathcal{A}_n}$, a.s., then $I_2^*\mathbb{1}_{\mathcal{A}_n} = (x - d_n)^+\mathbb{1}_{\mathcal{A}_n}$. Therefore $d_n = d_{n+1}$ as long as $\Pr\{\mathcal{A}_n\} > 0$. Hence there exists $d \geqslant 0$ such that $d_n = d, n >> 0$. So $I_2^*\mathbb{1}_{x > q} = (x - d)^+\mathbb{1}_{x > q}$.

Case 2. If there exists n such that $\Pr\{I_n \neq (x - d_n)^+ : \mathcal{A}_n\} > 0$, then $\Pr\{I_n = a + \frac{1}{n} : \mathcal{A}_n\} > 0$. Consider $I_{n+1} := \max\left\{a + \frac{1}{n+1}, (x - d_{n+1})^+\right\}$ over $\mathcal{A}_{n+1}$. Since I_{n+1} is non-decreasing, then $\Pr\left\{I_{n+1} = a + \frac{1}{n+1} : \mathcal{A}_{n+1}\right\} = 0$. Otherwise, I_{n+1} would start from a line $a + \frac{1}{n+1}$, and has another line in the middle $a + \frac{1}{n}$, which is impossible for a floored deductible indemnity (because of (C10) and of $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$). Therefore, $I_{n+1} = \max\left\{a + \frac{1}{n}, (x - d_n)^+\right\}$. Continuing this procedure we have

$$I_{n+m} = \max\left\{a + \frac{1}{n}, (x - d_n)^+\right\}, \forall m \geqslant 0.$$

Hence we see that, there exists n such that:

$$I_2^*\mathbb{1}_{x > q} = \max\left\{a + \frac{1}{n}, (x - d_n)^+\right\}.$$

By combining Step 2 and Step 3 together, we have either $I^* = \min\{a, (x - d_1)^+\}\mathbb{1}_{x \leqslant q} + \max\{f, (x - d_2)^+\}\mathbb{1}_{x > q}$, $f > a$ or $I^* = \min\{a, (x - d_1)^+\}\mathbb{1}_{x \leqslant q} + (x - d_2)^+\mathbb{1}_{x > q}$. Then by using lemmas C.2 and C.3 together, the optimal indemnity is given by

$$I^* = \min\left\{a, (x - d^*)^+\right\}\mathbb{1}_{x \leqslant q} + (x - d^*)^+\mathbb{1}_{x > q} \tag{C11}$$

where d^* is defined in lemma C.2. Then we find out one non-decreasing optimal indemnity based on this characterization. The uniqueness follows from the proof procedure.

The proof of Proposition 5.1 is then completed. $\qquad\square$

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