Improving the Design of Financial Products in a Multidimensional Black-Scholes Market

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Abstract

Using various techniques, authors have shown that in one-dimensional markets, complex (path-dependent) contracts are generally not optimal for rational consumers. In this paper we generalize these results to a multidimensional Black-Scholes market. In such a market, we discuss optimal contracts for investors who prefer more to less and have a fixed investment horizon $T > 0$. First, given a desired probability distribution, we give an explicit form of the optimal contract that provides this distribution to the consumer. Second, in the case of risk-averse investors, we are able to propose two ways of improving the design of financial products. In all cases, the optimal payoff can be seen as a path-independent European option that is written on the so-called market portfolio. We illustrate the theory with a few well-known securities and strategies. For example, we show that a buy-and-hold investment strategy can be dominated by a series of power options written on the underlying market portfolio. We also analyze the inefficiency of a widely used portfolio insurance strategy called Constant Proportion Portfolio Insurance.

1. Introduction

Equity-linked contracts that combine life-contingent benefits with financial benefits are popular in many insurance markets. They propose mortality benefits and capital protection with the potential for high returns if the market performs well. In a typical equity-indexed annuity, the consumer pays an initial amount to the insurance company. At the maturity date, the payoff to the investor is usually linked in a complicated way to the performance of some designated reference indices. In North America, the two main different types of equity-linked products sold by insurance companies are *variable annuities* and *equity-indexed annuities*. Insurance companies also use these complex payoff profiles in the management of their own assets and liabilities. For example a so-called Constant Proportion Portfolio Insurance (CPPI) strategy, which aims to protect the value of the company's investment portfolio while benefiting from increasing markets, gives rise to payoffs that are of the same complexity as equity-indexed annuity payoffs.

Dybvig (1988a,b) introduced an interesting concept to compare strategies without having to refer explicitly to the individual's preferences. With a focus on the *one-dimensional* market he showed that there could be several strategies with the same distribution at maturity but with different initial costs. A strategy is cost efficient if it is not possible to find another strategy that provides the same distribution of wealth at maturity at a strictly lower cost. The key result is that among all payoffs that have the same terminal wealth distribution, there is one path-independent payoff that will have a strictly lower cost than all others. Clearly this payoff will be preferred by all *profit-seeking* investors. For more details, see Dybvig (1988a,b) and Bernard and Boyle (2010), who provide an explicit representation of the optimal payoff to achieve a given distribution. Using other techniques, Cox and Leland (1982, 2000) showed that in a one dimensional Black-Scholes market path-dependent strategies are never

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optimal for *risk-averse* investors who have a fixed investment maturity. Based on an idea of Vanduffel (2005, p. 27), Vanduffel et al. (2009) extended the latter study to one-dimensional Lévy markets. In this paper, we extend some of these results to a *multidimensional* Black-Scholes market, that is, when the price processes are governed by correlated geometric Brownian motions. In particular, we make the following contributions.

First, for any given payoff we provide an explicit characterization of a new payoff that will be preferred by *all* profit-seeking decision makers in a multidimensional Black-Scholes market. This dominating payoff can be interpreted as a European path-independent option that is written on the so-called market portfolio.

Second, we propose an alternative way to build a European path-independent contract suitable for risk-averse investors.

Third, for any given risk-averse Von Neuman–Morgenstern decision maker we construct explicitly an optimal path-independent payoff. In this sense we complement the analysis of Merton (1969), whose analysis showed (among many other results) that in the case of an investor with constant relative risk aversion (a so-called CRRA investor) the optimal payoff effectively consists of purchasing a particular European power option written on the market portfolio.

Fourth, we analyze the efficiency of several payoffs. We briefly discuss a payoff obtained by pursuing a buy-and-hold strategy of the available assets and show that this strategy is never optimal in a multidimensional Black-Scholes market. We explicitly derive the dominating payoff from the point of view of all risk-averse investors, and it turns out that it can be interpreted as a series of power options written on the market portfolio. We also analyze CPPI strategies and discuss Margrabe options.

To summarize, in this paper we essentially show that in a multidimensional Black-Scholes market path-dependent payoffs are suboptimal, and we provide explicit characterizations of dominating (path-independent) payoffs. The suboptimality of path-dependent claims is valid only if the state-price process in the financial market is itself path independent and continuously distributed (as it is the case in our setting). In other words, our results do not necessarily hold in a discrete-time market model or in the presence of stochastic interest rates or stochastic volatility (which would make the state-price process path dependent).

The paper is structured as follows. In Section 2 we describe the multidimensional Black-Scholes market and the market portfolio, and we explain how to price in a multidimensional Black-Scholes market. Next, in Section 3 we analyze the efficiency of payoffs. In Section 4 we illustrate the theoretical results, and Section 5 concludes.

2. MULTIDIMENSIONAL BLACK-SCHOLES MARKETS

In this section we recall what a multidimensional Black-Scholes market is, how the "market portfolio" is defined, and how it relates to the optimal portfolio for a mean variance investor. Finally we explain how the market portfolio can be used to price in the multidimensional Black-Scholes market. Most results in this section are quite standard and can be found, for instance, in Dhaene et al. (2005) and Björk (2004).

2.1 Description

Consider the classical continuous-time framework that was pioneered by Merton (1969, 1971) and is nowadays mostly referred to as the Black and Scholes (1973) setting. To this end, let us assume that the market is frictionless, trading is continuous, and there is a constant continuously compounded risk-free interest $r > 0$. There are also no taxes, no transaction costs, no dividends, and no restriction on borrowing or short sales, and the m risky assets are perfectly divisible. Let $\mathbb{P}$ denote the physical probability measure. Let $S^i(0) > 0$ be the current price, at time 0, of one unit of the risky asset i, and $S^i(t)$ is its price at time t. Assume that the prices $S^i(t)(i = 1, 2, \ldots, m)$ are geometric Brownian motions.

They evolve according to the following stochastic differential equations:

$$\frac{dS^i(t)}{S^i(t)} = \mu_i dt + \sigma_i\, dB^i(t), \quad i = 1, 2, \ldots, m. \tag{1}$$

The m-dimensional vector $\mu^T = (\mu_1 \ldots \mu_m)$ is called the drift vector of the risky assets. We will assume that $\mu \neq r\,\mathbf{1}$, with $\mathbf{1}^T = (1\ 1 \ldots 1)$. Furthermore the processes $\{B^i(t), t \geq 0\}$, $i = 1, \ldots, m$ are (correlated) standard Brownian motions, with constant correlation coefficients ρ_{ij}:

$$\forall t, s \geq 0, \quad \rho_{ij} = \mathrm{Corr}(B^i(t), B^j(t+s)). \tag{2}$$

Furthermore, we also define a $(m \times m)$ matrix Σ as

$$\sum = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{m1} & \sigma_{m2} & \ldots & \sigma_m^2 \end{pmatrix}, \tag{3}$$

where $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_i$. Note that $\sigma_{ij} = \sigma_{ji}$ and that $\sigma_{ii} = \sigma_i^2$, and it is also assumed that Σ is positive definite. It is well known that the solution to equation (1) for $i = 1, \ldots, m$ is

$$S^i(t) = S^i(0)\exp(X^i(t)), \quad t > 0, \tag{4}$$

where

$$X^i(t) = \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i B^i(t). \tag{5}$$

Also it is easily seen that the elements of the matrix Σ describe the covariances between the yearly log-returns $X^i(k) - X^i(k-1)$ of the m assets $(i = 1, 2, \ldots, m)$, and k is a positive integer.

2.2 The Market Portfolio

We first recall the definition of a constant mix strategy. It is later needed to derive the market portfolio and to characterize optimal strategies. A strategy at time t is denoted by $\pi(t)^T = (\pi_1(t), \pi_2(t), \ldots, \pi_m(t))$, where $\pi_i(t)$ is the fraction of the wealth that is invested in risky asset i at time t. The residual, $1 - \Sigma_{i=1}^n \pi_i(t)$, is invested in the risk-free asset, which grows at the constant continuously compounded interest rate r or, if negative, finances the risky asset purchases. A constant portfolio $\pi(t) = \pi = (\pi_1, \pi_2, \ldots, \pi_m)^T$, where the fractions invested in the different assets remain constant over time, is called a constant mix strategy.

It follows that under the $\mathbb{P}$-measure the dynamics of the price process $\{S^\pi(t), t \geq 0\}$ of the security that is constructed according to a nonzero vector π is given by

$$\begin{aligned} \frac{dS^\pi(t)}{S(t)} &= \sum_{i=1}^m \pi_i \frac{dS^i(t)}{S^i(t)} + \left(1 - \sum_{i=1}^m \pi_i\right) rdt \\ &= \left(\sum_{i=1}^m \pi_i(\mu_i - r) + r\right) dt + \sum_{i=1}^m \pi_i\sigma_i dB^i(t). \end{aligned} \tag{6}$$

It is easy to see that (6) can be recast as

$$\frac{dS^\pi(t)}{S(t)} = \mu(\pi)dt + \sigma(\pi)dB^\pi(t), \tag{7}$$

where $\{B^\pi(t), t \geq 0\}$ is a standard Brownian motion defined through

$$B^\pi(t) = \frac{1}{\sqrt{\pi^T \cdot \Sigma \cdot \pi}} \sum_{i=1}^m \pi_i\sigma_i B^i(t), \tag{8}$$

with $\mu(\pi)$ and $\sigma^2(\pi)$ given by

$$\mu(\pi) = r + \pi^T \cdot (\mu - r\mathbf{1}), \text{ and } \sigma^2(\pi) = \pi^T \cdot \Sigma \cdot \pi, \tag{9}$$

respectively, and where $\mathbf{1}$ is the m-vector $(1\ 1\ \ldots\ 1)^T$. Recall that the solution to equation (7) is

$$S^\pi(t) = S^\pi(0) \exp(X^\pi(t)), \tag{10}$$

with

$$X^\pi(t) = \left(\mu(\pi) - \frac{1}{2}\sigma^2(\pi)\right) t + \sigma(\pi) B^\pi(t). \tag{11}$$

The following result is well known; see also Dhaene et al. (2005).

Proposition 1 (Market Portfolio)

Assume that $\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1}) > 0$ and $\sigma > 0$. The solution of the following mean-variance optimization problem

$$\max_\pi \mu(\pi) \text{ subject to } \sigma(\pi) = \sigma \tag{12}$$

is denoted by π^σ and is given by

$$\pi^\sigma = \frac{\sigma}{\sqrt{(\mu - r\mathbf{1})^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1})}} \Sigma^{-1} \cdot (\mu - r\mathbf{1}). \tag{13}$$

Hence, for the appropriate choice of σ, there will be a unique mean-variance efficient portfolio π^σ fully invested in the risky assets ($\mathbf{1} \cdot \pi^\sigma = 1$). We call it the "market portfolio" and denote it by π^*:

$$\pi^* = \frac{\Sigma^{-1} \cdot (\mu - r\mathbf{1})}{\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1})}. \tag{14}$$

From (9) it follows then that $\mu_* := \mu(\pi^*)$ and $\sigma_*^2 := \sigma^2(\pi^*)$ are now given by

$$\mu_* = r + \frac{(\mu - r\mathbf{1})^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1})}{\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1})} \tag{15}$$

and

$$\sigma_*^2 = \frac{(\mu - r\mathbf{1})^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1})}{(\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1}))^2}. \tag{16}$$

Note that from (8) and(14) it also follows that

$$\begin{aligned} \mathrm{Cov}[\sigma_* B^{\pi^*}(t), \sigma_i B^i(t)] &= \left(\frac{\Sigma^{-1} \cdot (\mu - r\mathbf{1})}{\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1})}\right)^T \cdot \Sigma \cdot \mathbf{1}^i t \\ &= \frac{\mu_i - r}{\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1})} t, \end{aligned} \tag{17}$$

where $\mathbf{1}^i$ is used to denote a $m \times 1$ vector with value 1 in the ith row and zero elsewhere.

The label "market portfolio" will also be used to refer to the security with price process $\{S^{\pi^*}(t), t \geq 0\}$. Note that the condition $\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1}) > 0$ makes sense economically. Indeed, it can be shown that the so-called minimal variance portfolio π^m, which is the solution of the problem

$$\mathrm{Min}_\pi \sigma(\pi) \text{ subject to } \mathbf{1}^T \cdot \pi = 1, \tag{18}$$

has a drift $\mu(\pi^m)$ that is given by

$$\mu(\pi^m) = \frac{\mathbf{1}^T \cdot \Sigma^{-1} \cdot \mu}{\mathbf{1}^T \cdot \Sigma^{-1} \cdot \mathbf{1}}, \tag{19}$$

so that $\mu(\pi^m) > r$ if and only if $\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1}) > 0$, suggesting that in financial markets the condition $\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1}) > 0$ should indeed hold (see also Dhaene et al. 2005 for further interpretation).

Finally, for any constant mix strategy π the coefficient $\theta(\pi)$ defined as

$$\theta(\pi) = \frac{\mu(\pi) - r}{\sigma(\pi)} \tag{20}$$

is called the market price of risk or Sharpe ratio of the constant mix portfolio π. In the remainder of the paper we will always assume that the condition $\mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1}) > 0$ is fulfilled. Then it follows from (15) that

$$\theta_* := \theta(\pi^*) > 0. \tag{21}$$

2.3 Pricing in a Multidimensional Black-Scholes Market

It is well known that the multidimensional market described above is arbitrage free and complete. Let $\{\mathcal{F}_t, t \geq 0\}$ be the natural filtration generated by the m assets prices $S^i(t)(i = 1, 2, \ldots, m)$ and let us recall that the market is equipped with a physical probability measure $\mathbb{P}$. We now define the state-price process.

Definition 1 (State-Price Process)

If the stochastic process $\{\xi(t), t \geq 0\}$ is such that for each $i = 1, 2, \ldots, m$ the process $\{\xi(t)S^i(t), t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ and the measure $\mathbb{P}$, then we call $\{\xi(t), t \geq 0\}$ a state-price process.

Consider a strategy with a $\mathcal{F}_T$-measurable payoff $H(T)$ paid at time T. Its initial price is equal to

$$c(H(T)) = \mathrm{E}_{\mathbb{P}}[\xi(T)H(T)], \tag{22}$$

where $E_{\mathbb{P}}$ is the expectation under the physical probability measure $\mathbb{P}$. We assume that $c(H(T))$is well defined and finite. The following proposition provides a state-price process $\{\xi(t), t \geq 0\}$ for the multidimensional Black-Scholes market as outlined above.

Proposition 2 (State-Price Process for a Black-Scholes Market)

The process $\{\xi(t), t \geq 0\}$ with $\xi(t) = e^{-rt}e^{\theta_*^2/2t-\theta_* B^{\pi^*}(t)}$ is the state-price process where π^* corresponds to the market portfolio (given by [14]). Then the state price $\xi(t)$ *can be written as an explicit function of the market portfolio price $S^{\pi^*}(t)$*:

$$\xi(t) = a \cdot \left(\frac{S^{\pi^*}(t)}{S^{\pi^*}(0)}\right)^{-b}, \tag{23}$$

where $a = \exp\,(\theta_*/\sigma_*(\mu_* - \sigma_*^2/2)t - (r + \theta_*^2/2)t)$ and $b = \theta_*/\sigma_*$.

PROOF

The proof of Proposition 2 is given in the Appendix. Similar concepts are discussed in sections 14.6 and 14.7 of Björk (2004) and in Duffie (1992). ■

Proposition 2 gives the link between the state-price process and the market portfolio. This link is going to be a key result toward the optimality of financial payoffs.

The state-price process is also tied to the change of measure from the historical probability $\mathbb{P}$ to the unique risk-neutral probability $\mathbb{Q}$ (in the arbitrage-free and complete multidimensional Black-Scholes market). Indeed, it is well known that the initial price of a financial derivative can also be seen to be the expectation of the discounted payoff under the risk-neutral probability; that is, the initial price of the payoff $H(T)$ can also be expressed as

$$c(H(T)) = \mathrm{E}_{\mathbb{Q}}[e^{-rT}H(T)]. \tag{24}$$

Comparing (22) and (24), it is then clear that $\xi(T) = e^{-rT}\, d\mathbb{Q}/d\mathbb{P}$; in particular we have that

$$E_{\mathbb{Q}}[e^{-rT}H(T)] = E_{\mathbb{P}}[\xi(T)H(T)]. \tag{25}$$

We refer to Harrison and Kreps (1979), Harrison and Pliska (1981), and Björk (2004) for extensive theory on arbitrage-free pricing.

3. Optimal Payoffs

To improve the design of financial products in a multidimensional market, we are going to extend the results from Bernard and Boyle (2010) and Vanduffel et al. (2009). To this end we first define the concept of cost efficiency.

Definition 2 (Cost Efficiency)

A strategy with payoff $H(T)$ at maturity T is cost efficient if it is not possible to construct another strategy (payoff) that generates the same distribution at T (under $\mathbb{P}$) but at a lower cost.

This concept was first introduced by Dybvig (1988a,b). In the following, we present Theorem 1 for profit-seeking investors and Theorem 2 for risk-averse investors. Both results explain how, starting from a strategy generating a payoff $H(T)$ that is not a function of the market portfolio at time T, it is possible to strictly improve the "design" of $H(T)$ from the investor's perspective by considering some strategies that directly depend on the market portfolio.

3.1 The Case of Profit-Seeking Investors

Consider a random variable X with distribution F, where F is right-continuous and nondecreasing. We define its inverse F^{-1} as

$$F^{-1}(y) = \min\,\{x \mid F(x) \geq y\}.$$

Theorem 1 (Given Payoff ⇒ Cheapest Payoff with the Same Distribution)

Consider a payoff $H(T)$ with distribution function F (under $\mathbb{P}$), which is assumed to be strictly increasing. Consider the payoff $Z^*(T)$ given as

$$Z^*(T) = F^{-1}(F_{S^{\pi^*}(T)}(S^{\pi^*}(T))). \tag{26}$$

Then the payoff $Z^*(T)$ will have F as its $\mathbb{P}$-distribution function. Further, it will hold that

$$c(Z^*(T)) \leq c(H(T)). \tag{27}$$

Finally, $Z^*(T)$ is the almost surely unique way to achieve the cheapest payoff with distribution F at time T.

Proof

This follows from Theorem 1 of Bernard and Boyle (2010), which for general markets relates cost efficiency to the state price $\xi(T)$, and Proposition 2, which expresses the state price $\xi(T)$ as an explicit function of the market portfolio price $S^{\pi}_{*}(T)$. See the Appendix for a more detailed proof. ■

This result is consistent with the previous literature on this topic. Actually, expression (26), characterizing cost-efficient payoffs for a given distribution, can also be traced back to Dybvig (1988a, Theorem 1), who proved the optimality of $Z^*(T)$ in the context of a *complete one-dimensional* market. More recently Vanduffel et al. (2009) extended Dybvig's result to *incomplete one-dimensional* Lévy markets. Here we characterize optimal payoffs in *multidimensional* Black-Scholes markets.

Theorem 1 has important consequences in terms of optimal investment strategies for profit-seeking investors with a fixed investment horizon. More specifically, for a fixed initial budget (cost), let $X(T)$ be the final wealth of the investor resulting from an investment strategy. Let us then assume that the objective function of the investor, $V(X(T))$, satisfies the three following properties:

- The investor has a fixed investment horizon, and there is no intermediate consumption.[1]
- The investor prefers "more to less," that is, V preserves first-order stochastic dominance relationships (denoted by $<_{fsd}$). If $Y(T) <_{fsd} X(T)$, then $V(X(T)) \geq V(Y(T))$.
- The investor has "state-independent preferences" or "law-invariant preferences": If $Z(T)$ has the same distribution as $W(T)$, then $V(Z(T)) = V(W(T))$.

Under these three fairly general assumptions, and taking into account the initial cost of purchasing a payoff, it follows immediately that the cost-efficient payoff will be preferred to any other payoff that has the same distribution. The stated assumptions on agents' preferences are similar to those used by Dybvig (1988a,b). Such preferences include a wide range of decision theories, among them the expected utility theory (Von Neuman and Morgenstern 1947), Yaari's dual theory of choice under risk (Yaari 1987), and more recent behavioral theories such as the cumulative prospect theory (Tversky and Kahneman 1992) and the rank-dependent utility theory (Quiggin 1993).

Finally, note that the dominating payoff $Z^*(T)$ from Theorem 1 can be seen as a European path-independent derivative written on the market portfolio. Therefore the theorem essentially states that for a given payoff $H(T)$ one can find a path-independent alternative $Z^*(T)$ that will be preferred by *all* profit-seeking decision makers, also meaning that no utility function needs to be specified to construct the dominating alternative.

3.2 The Case of Risk-Averse Investors

Cox and Leland (1982, 2000) employed stochastic control theory to show that in *one-dimensional* Black-Scholes markets risk-averse decision makers with a fixed investment horizon prefer path-independent payoffs over path-dependent payoffs. Vanduffel et al. (2009) extended these results to *one-dimensional* Lévy markets. The corresponding results holding for multidimensional Black-Scholes markets did not appear yet in the literature.

Theorem 2 (A Given Payoff ⇒ Less Risky Payoff at the Same Cost)

Let $H(T)$ be the payoff of a given strategy. Define

$$H^*(T) = E_{\mathbb{P}}[H(T) \mid S^{\pi^*}(T)]. \tag{28}$$

Then $H^*(T)$ has the same cost as $H(T)$:

$$c(H^*(T)) = c(H(T)), \tag{29}$$

and $H^*(T)$ dominates $H(T)$ in the sense of second-order stochastic dominance.

Proof

Note that for the sake of simplicity, we write $E[X|Y]$ instead of $E[X|\sigma(Y)]$, where $\sigma(Y)$ denotes the σ-field generated by the random variable Y. From the relationship (23) between $\xi(T)$ and $S^{\pi^*}(T)$, it is

[1] Note that our results can be generalized to the case when T is random if the random maturity time is independent from the financial market. It could be the time of death of the investor, for example. Proofs should then be slightly modified, and one has to use conditioning with respect to τ first.

clear that $\sigma(\xi(T)) = \sigma(S^{\pi^*}(T))$, and therefore it is also clear that the payoff $H^*(T)$ can also be expressed as $H^*(T) = E_{\mathbb{P}}\ [H(T)|\xi(T)]$. The fact that $H^*(T)$ has the same cost as $H(T)$ now follows from the definition of the initial price (22) and the tower property for expectations directly. Formally:

$$\begin{aligned} c(H(T)) &= E_{\mathbb{P}}[\xi(T)H(T)] \\ &= E_{\mathbb{P}}[E_{\mathbb{P}}[\xi(T)H(T)|\xi(T)]] \\ &= E_{\mathbb{P}}[\xi(T)E_{\mathbb{P}}[H(T)|\xi(T)]] \\ &= c(H^*(T)). \end{aligned} \tag{30}$$

Finally, the fact that the payoff $H^*(T)$ dominates $H(T)$ in the sense of second-order stochastic dominance is well known and is a direct application of Jensen's famous inequality. ■

The theorem has some important implications for the class of risk-averse decision makers. Indeed, for a fixed initial budget (cost), let $X(T)$ be the final wealth (payoff) of the investor resulting from a particular investment strategy. We assume that the objective function of the investor, $V(X(T))$, now satisfies the following three properties:

- The investor has a fixed investment horizon, and there is no intermediate consumption.
- The investor is ''risk averse,'' that is, V preserves second-order stochastic dominance relationships. If $Y(T) \prec_{ssd} X(T)$, then $V(X(T)) \geq V(Y(T))$.
- The agent has ''state-independent preferences'' or ''law-invariant preferences'': If $Z(T)$ has the same distribution as $W(T)$, then $V(Z(T)) = V(W(T))$.

Since we have found that $c(H^*(T)) = c(H(T))$, it follows directly that $H^*(T) - c(H^*(T))$ dominates $H(T) - c(H(T))$ in the second-order stochastic dominance. Hence our investor will always prefer the ''less risky'' payoff $H^*(T)$ above the original payoff $H(T)$. Or to put it differently, the payoff $H^*(T)$is at least as good as $H(T)$ for all risk-averse investors.

Note that the stated preferences are those of risk-averse decision makers in expected utility theory (Von Neuman and Morgenstern 1947) and Yaari's dual theory of choice under risk (Yaari 1987).

We also remark that this dominating payoff $H^*(T)$ is by definition of the conditional expectation a measurable function of the market portfolio $S^{\pi^*}(T)$ and hence can also be seen as a European path-independent derivative written on the market portfolio. Therefore the theorem states that for a given payoff $H(T)$ one can find a path-independent alternative $H^*(T)$ that will be preferred by *all* risk-averse decision makers, and no utility function needs to be specified to construct $H^*(T)$. It is then also clear that Theorems 1 and 2 both provide ways to improve the design of financial derivatives by having a strategy whose payoff simply depends on the value of the market portfolio at maturity T (which therefore is a path-independent payoff).

However, it is only when $H^*(T)$ is an increasing function of $S^{\pi^*}(T)$ that it will lead to a cost-efficient contract in the sense of definition 2. In fact, the following remark can be made.

REMARK 1 (CONNECTIONS BETWEEN $H^*(T)$ AND $Z^*(T)$)

Consider the final payoff $H(T)$ of the contract, as well as the dominating alternatives $Z^*(T)$ from Theorem 1 and $H^*(T)$ from Theorem 2, respectively. Then it holds that

- *$Z^*(T)$ and $H^*(T)$ are different unless the original payoff $H(T)$ is an increasing function of $S^{\pi^*}(T)$. In this case $X(T) = H^*(T) = Z^*(T)$ a.s.*
- *$Z^*(T)$ has the same distribution as $H(T)$ but a lower cost, whereas $H^*(T)$ has a possibly different distribution from $H(T)$ but the same initial cost.*
- *In general $H^*(T)$ is not cost efficient. It is cost efficient if and only if it is an increasing function of $S^{\pi^*}(T)$.*

Regarding the optimality of single-asset payoffs, the following remark can be made.

REMARK 2 (SUBOPTIMALITY OF ANY SINGLE-ASSET PAYOFF)

In a multidimensional Black-Scholes market ($m > 1$) *any single-asset payoff* $H(T) = h(S^i(t)_{[0<t\leq T]})$ *with a given* $i \in \{1, 2, \ldots, m\}$ *is dominated by* $E_{\mathbb{P}}[H(T)|S^{\pi^*}(T)$. *In particular, even while* $H(T)$ *could be optimal for a risk-averse decision maker in the context of a one-dimensional Black-Scholes market governed by the risky asset* $S^i(t)$, *it is suboptimal in the given multidimensional market.*

From the remark we conclude that in a one-dimensional Black-Scholes market one can find for a given payoff $H(T)$ a new payoff $E_{\mathbb{P}}[H(T)|S^1(T)]$, which is robust in the sense that all risk-averse decision makers, irrespective of the precise form of the concave utility function they use and irrespective of their subjective choices for the drift parameter μ_1, will prefer $E_{\mathbb{P}}[H(T)|S^1(T)]$ above $H(T)$. Note that this cannot be readily generalized to the multidimensional case directly because the weights π_i^* ($i = 1, 2, \ldots, m$) depend on the μ_i (see [14]).

3.3 Optimal Payoffs

While Theorem 1 explains how to construct the cheapest strategy to achieve a given distribution F, Theorem 2 implies that only path-independent payoffs are relevant to the risk-averse decision makers, but it does not tell us which payoffs are optimal for a given investor.

Given specific preferences of the investor, there is an optimal final wealth and therefore an optimal probability distribution at maturity. In this section, for a given risk-averse Von Neuman–Morgenstern decision maker (with a specific utility function), we give an explicit form of the ''best'' path-independent payoff. We then verify that it is cost efficient and that it cannot be improved.

Theorem 3 (Risk-Averse Decision Maker ⇒ Optimal Payoff)

Consider a risk-averse decision maker with utility function $x \to u(x)$ *and initial wealth* W_0. *Let us assume that* $u(x)$ *is continuously differentiable with strictly decreasing derivative* $u'(x)$. *Then we find that the optimal payoff* $H^*(T)$ *is given by*

$$H^*(T) = h^*(S^{\pi^*}(T)) \tag{31}$$

with $x \to h^*(x)$ *given by*

$$h^*(x) = (u')^{-1}(c \cdot x^{-\theta_*/\sigma_*}), \tag{32}$$

where $\theta_* = \mu_* - r/\sigma_*$, *and* $c > 0$ *is such that it holds that*

$$W_0 = E_{\mathbb{P}}[\xi(T)h^*(S^{\pi^*}(T))]. \tag{33}$$

PROOF

The proof is given in the Appendix. ■

Theorem 3 allows us to describe the optimal portfolio for any risk-averse decision maker in a multidimensional Black-Scholes market. The optimal payoff $H^*(T)$ will be a function of the terminal value $S^{\pi^*}(T)$ of the market portfolio π^*, and we provide the link function $x \to h^*(x)$ explicitly. Note that the concavity of u and the positivity of θ_* imply that h^* is indeed an increasing function of $S^{\pi^*}(T)$, and therefore from Theorem 1, it is cost efficient.

We are now ready to derive Merton's optimal solution as a special case (see Merton 1969).

EXAMPLE 1 (MERTON'S OPTIMAL PORTFOLIO FOR THE CRRA INVESTOR)

Let us assume that the investor is exhibiting constant relative risk aversion (CRRA investor), that is, we have that

$$\begin{aligned} u(x) &= \frac{x^{1-\gamma}}{1-\gamma} \quad \textit{for } \gamma > 0, \gamma \neq 1 \\ &= \log(x) \quad \textit{for } \gamma = 1. \end{aligned} \tag{34}$$

Because $u'(x) = x^{-\gamma}$ it follows from Theorem 3 that

$$H^*(T) = c^{-1/\gamma} \cdot (S^{\pi^*}(T))^{1/\gamma\,\theta_*/\sigma_*}, \tag{35}$$

where c is the appropriate constant[2] *chosen such that*

$$W_0 = E_{\mathbb{P}}[\xi(T)H^*(T)] \tag{36}$$

holds. Hence it follows that $H^*(T)$ is the payoff of a constant mix strategy characterized by π^*_α:

$$H^*(T) = W_0 \cdot \exp(X^{\pi^*_\alpha(T)}). \tag{37}$$

Here $\pi^*_\alpha = \alpha\pi^*$ with α given by

$$\alpha = \frac{1}{\gamma}\frac{\theta_*}{\sigma_*}. \tag{38}$$

PROOF

The proof of (37) is shown in the Appendix. ■

Example 1 shows that all CRRA investors apply a constant mix strategy where they invest a proportion equal to $\alpha = 1/\gamma\ \theta_*/\sigma_*$ in the market portfolio π^* and the remaining part $1 - \alpha$ in the risk-free asset, and note that this proportion α will be inversely related to the investor's level of risk aversion γ; see Merton (1969). Note that $H^*(T)$ can also be considered as the payoff of a European power call option written on the underlying market portfolio with zero strike.

4. ILLUSTRATIONS

We illustrate the study with three examples. First, we discuss buy-and-hold strategies and illustrate how inefficient they are and how they can be dominated. We then discuss why Margrabe exchange options are not optimal and how better contracts can be constructed. Finally a popular portfolio insurance strategy is investigated.

4.1 Buy-and-Hold Strategies

In this section we will discuss the inefficiency of a buy-and-hold strategy and show that it can be dominated by purchasing a series of power options with zero strike. To this end let us consider the payoff $H(T)$ given by

$$H(T) = W_0\left(\sum_{i=1}^{m}\alpha_i e^{X^i(T)} + \left(1 - \sum_{i=1}^{m}\alpha_i\right)e^{rT}\right), \tag{39}$$

where the random variables $X^i(T)$ are representing the $N((\mu_i - \sigma_i^2/2)T,\ \sigma_i^2 T)(i = 1, 2, \ldots, m)$ distributed log-returns over the period $[0,T]$. The payoff $H(T)$ can be seen as the result of a buy-and-hold strategy evaluated at the end of the investment horizon T, where at time $t = 0$, one invests α_i in the ith risky asset and $1 - \sum_{i=1}^m \alpha_i$ in the riskless asset and where one does not trade afterwards.

In general the distribution of $H(T)$ is a sum of lognormal distributions, and it is not a well-known distribution. Therefore we will not get a closed-form expression of the cost-efficient payoff $Z^*(T)$ obtained in Theorem 1, although we could approximate it by way of Monte Carlo techniques. It is interesting to see that the conditional expectation defined in Theorem 2 can be explicitly calculated. Hence let us consider next the path-independent payoff $H^*(T) := E_{\mathbb{P}}[H(T)|X^{\pi^*}(T)]$. Note that when $Y = e^X$ follows a lognormal distribution, $E[Y|\mathcal{F}]$ is simply $\exp(E[X|\mathcal{F}] + \frac{1}{2}\,Var[X|\mathcal{F}])$. Therefore using properties of multivariate normal distributions we find that $H^*(T)$ is equal to

[2] In practice, this constant can be obtained numerically.

$$H^*(T) = W_0 \cdot \sum_{i=1}^{m} \alpha_i e^{\mathbb{M}_i + 1/2\mathbb{V}_i} + W_0 \cdot \left(1 - \sum_{i=1}^{m} \alpha_i\right) e^{rT}, \tag{40}$$

where $\mathbb{M}_i = E_{\mathbb{P}}[X^i(T) | X^{\pi^*}(T)]$ and $\mathbb{V}_i = Var_{\mathbb{P}}[X^i(T) | X^{\pi^*}(T)]$. For $i \in \{1, 2, \ldots, m\}$, one has

$$\begin{cases} \mathbb{M}_i = E_{\mathbb{P}}[X^i(T)] + \dfrac{Cov_{\mathbb{P}}[X^i(T), X^{\pi^*}(T)]}{Var_{\mathbb{P}}[X^{\pi^*}(T)]} (X^{\pi^*}(T) - E_{\mathbb{P}}[X^{\pi^*}(T)]), \\ \mathbb{V}_i = Var_{\mathbb{P}}[X^i(T)] - \dfrac{Cov^2_{\mathbb{P}}[X^i(T), X^{\pi^*}(T)]}{Var_{\mathbb{P}}[X^{\pi^*}(T)]}. \end{cases} \tag{41}$$

Hence, using the notation introduced before we find that the buy-and-hold payoff $H(T)$ can be dominated by $H^*(T)$:

$$H^*(T) = W_0 \sum_{i=1}^{m} \alpha_i e^{b_i} (e^{X^{\pi^*}(T)})^{c_i} + W_0 \left(1 - \sum_{i=1}^{m} \alpha_i\right) e^{rT}, \tag{42}$$

with constants b_i and c_i ($i = 1, 2, \ldots, m$) given by

$$b_i = \left(\left(\mu_i - \frac{\sigma_i^2}{2}\right) - \rho_i \frac{\sigma_i}{\sigma_*}\left(\mu_* - \frac{\sigma_*^2}{2}\right) + \frac{1}{2}(1 - \rho_i^2)\sigma_i^2\right) T, \tag{43}$$

and

$$c_i = \rho_i \frac{\sigma_i}{\sigma_*}, \tag{44}$$

respectively, where $\rho_i = \mathrm{Corr}_{\mathbb{P}}[X^i(T), X^{\pi^*}(T)]$. Note that b_i can also be determined from

$$b_i = \left(r - \rho_i \frac{\sigma_i}{\sigma_*}\left(r - \frac{\sigma_*^2}{2}\right) - \frac{1}{2}\rho_i^2\sigma_i^2\right) T. \tag{45}$$

Hence the optimal payoff $H^*(T)$ depends only on the drift parameters $\mu_i (i = 1, 2, \ldots, m)$ through the weights π_i^* of the market portfolio.

By substituting in expression (42) the component $e^{X^{\pi^*}(T)}$ by $S^{\pi^*}(T)/S^{\pi^*}(0)$, we see that $H^*(T)$ can also be interpreted as a weighted sum of power options with strike 0 written on the market portfolio.

As a special case, it is in general not optimal to hold an equally weighted portfolio with $1/m$ of the wealth in each asset. This portfolio is indeed dominated by the payoff (42) where $\alpha_i = 1/m$.

CASE WITH TWO ASSETS AND A RISKY PORTFOLIO

For example, let us take $T = 1$, $m = 2$, $W_0 = 1$, and $\alpha_1 + \alpha_2 = 1$. Further, we consider drift parameters $\mu_1 = 0.06$ and $\mu_2 = 0.10$, volatilities $\sigma_1 = 0.10$ and $\sigma_2 = 0.20$, and as correlation coefficient between the two risky assets we take $\rho = 0.5$. The risk-free rate r is equal to 0.03. From (14) we find that the market portfolio is determined by $\pi^* = (5/9, 4/9)$. Moreover, the log-return $X^{\pi^*}(1)$ has a drift $\mu_* = 7/90$ and a volatility $\sigma_* = 1/30\sqrt{43/3}$. We also find that $b_1 = 14/1075$, $c_1 = 27/43$, $b_2 = -5/258$, and $c_2 = 63/43$. Hence, the buy-and-hold payoff $H(1)$ can be dominated by the following payoff:

$$E_{\mathbb{P}}[H(1) | X^{\pi^*}(1)] = \alpha_1(e^{14/1075)(e^{X^{\pi^*}(1)}})^{27/43} + \alpha_2(e^{-5/258})(e^{X^{\pi^*}(1)})^{63/43}. \tag{46}$$

CASE OF A PORTFOLIO CONSTITUTED WITH ONE STOCK

As a special case, in a multidimensional Black-Scholes market investing in a portfolio with a single stock is not optimal:

$$H(T) = W_0(\alpha e^{X^1(T)} + (1 - \alpha)e^{rT}). \tag{47}$$

There are several ways to dominate this strategy with payoff $H(T)$ using Theorem 1 or Theorem 2. In our example, there are m assets, and the wealth is invested in the first risky asset and in the bond. Then it is possible to calculate explicitly the cost-efficient payoff (Theorem 1) as well as the conditional expectation (Theorem 2). We are going to calculate explicitly both and compare them graphically. Applying Theorem 1,we have that the cheapest way to achieve the distribution F of $H(T)$ is the payoff

$$Z^*(T) = F^{-1}(F_{S^{\pi^*}(T)}(S^{\pi^*}(T))).$$

Because $\ln(S^{\pi^*}(T)) = \ln(S^{\pi^*}(0)) + (\mu_* - \sigma_*^2/2)T + \sigma_* B^{\pi^*}(T)$, one has

$$\forall x > 0,\ \ F_{S^{\pi^*}(T)}(x) = \Phi\left(\frac{\ln\,(x/S^{\pi^*}(0)) - \left(\mu_* - \frac{\sigma_*^2}{2}\right)T}{\sigma_*\sqrt{T}}\right). \tag{48}$$

Let us calculate $F(x) = P(H(T) \le x)$ and its inverse. There are two cases. When $\alpha > 0$, the cdf of $H(T)$ is equal to

$$F(x) = \begin{cases} 0 & \text{if } x < W_0(1-\alpha)e^{rT}, \\ \Phi\left(\dfrac{\ln\left(\dfrac{x}{\alpha W_0} - \dfrac{(1-\alpha)e^{rT}}{\alpha}\right) - \left(\mu_1 - \dfrac{\sigma_1^2}{2}\right)T}{\sigma_1\sqrt{T}}\right) & \text{otherwise.} \end{cases} \tag{49}$$

Then, for $y \in (0, 1)$

$$F^{-1}(y) = \begin{cases} 0 & \text{if } y = 0 \\ (1-\alpha)e^{rT}W_0 + \alpha W_0 \exp\left(\Phi^{-1}(y)\sigma_1\sqrt{T} + \left(\mu_1 - \dfrac{\sigma_1^2}{2}\right)T\right) & \text{if } y > 0. \end{cases}$$

When $\alpha < 0$, then

$$F(x) = \begin{cases} 1 & \text{if } x > W_0(1-\alpha)e^{rT}, \\ \Phi\left(\dfrac{\ln\left(\dfrac{\alpha W_0}{x - (1-\alpha)W_0 e^{rT}}\right) + \left(\mu_1 - \dfrac{\sigma_1^2}{2}\right)T}{\sigma_1\sqrt{T}}\right) & \text{otherwise.} \end{cases} \tag{50}$$

Then, for $y \in (0, 1)$,

$$F^{-1}(y) = \begin{cases} (1-\alpha)W_0 e^{rT} & \text{if } y = 1 \\ (1-\alpha)e^{rT}W_0 + \alpha W_0 \exp\left(\left(\mu_1 - \dfrac{\sigma_1^2}{2}\right)T - \sigma_1\sqrt{T}\Phi^{-1}(y)\right) & \text{if } y = 1. \end{cases} \tag{51}$$

When $\alpha < 0$, $Z^*(T)$ is computed using the same expression as before, $Z^*(T) = F^{-1}(F_{S^{\pi^*}(T)}(S^{\pi^*}(T)))$, but where F^{-1} is calculated using the formula (51).

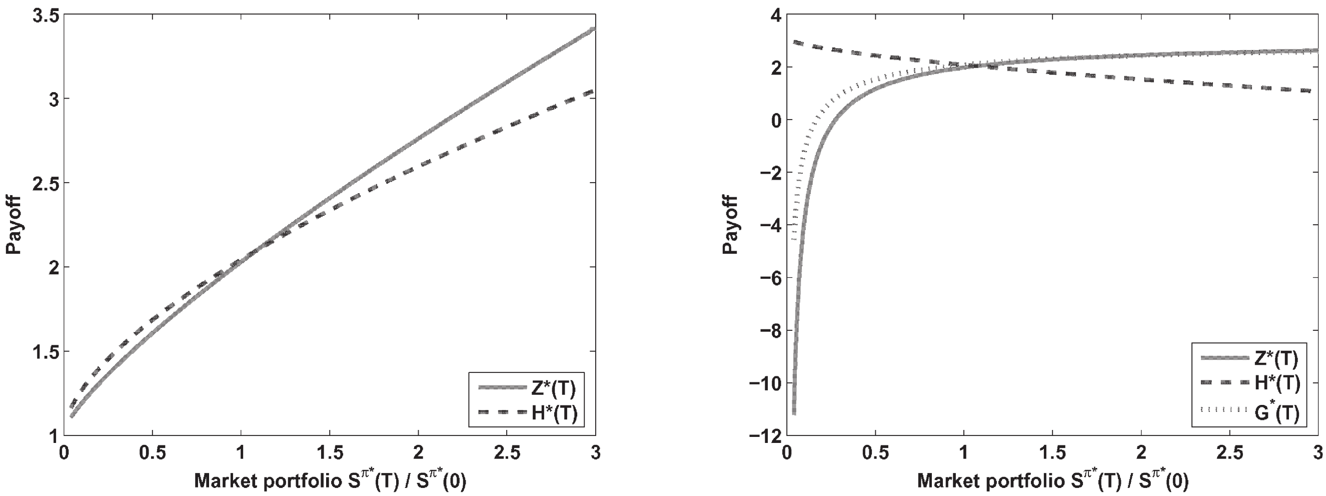
Applying Theorem 2, and using the expressions of c_1 and b_1 given by (43) and (44), one knows that

$$H^*(T) = W_0\alpha e^{b_1}(e^{X^{\pi^*}(T)})^{c_1} + W_0(1-\alpha)e^{rT}$$

dominates $H(T)$ for risk-averse investors and has the same cost as $H(T)$. This expression is valid for all $\alpha \in \mathbb{R}$.

When $\alpha > 0$, since $c_1 > 0$ it is cost efficient. In the case when $\alpha < 0$ it can be improved by applying Theorem 1, and the payoff

Figure 1
Comparison of Payoffs $Z^*(T)$, $G^*(T)$ and $H^*(T)$ for Buy-and-Hold Strategy with $\alpha = 0.5$ and $\alpha = -0.5$



Note: Parameters are $r = 3\%$, $\mu_1 = 0.06$, $\sigma_1 = 0.1$, $\mu_2 = 0.1$, $\sigma_2 = 0.2$, $\rho = 0.5$, $T = 1$ year, $W_0 = 2$. When $\alpha = 0.5$, we represent the two cost-efficient payoffs $H^*(T)$ and $Z^*(T)$ in the left panel. When $\alpha = -0.5$, we represent the cost-efficient payoff $G^*(T)$ and the inefficient payoff $H^*(T)$ in the right panel.

$$G^*(T) = \alpha W_0 e^{b_1} \left(\frac{S^{\pi^*}(T)}{S^{\pi^*}(0)}\right)^{-c_1} + W_0 e^{rT}(1 - \alpha) \tag{52}$$

dominates in the sense of first stochastic dominance of the payoff $H^*(T)$. The proof of (52) is given in the Appendix.

In Figure 1 we give a numerical illustration. Assume that the market has two assets and use the same parameters as in the previous example. In particular, $\sigma_* = 1/30\sqrt{43/3}$ and $\mu_* = 7.90$. In the left panel, we compare $Z^*(T)$ and $H^*(T)$ in the case when $\alpha > 0$. In the case when $\alpha < 0$, the payoffs $Z^*(T)$, $H^*(T)$, and $G^*(T)$ are represented in the right panel.

4.2 Margrabe Option

For the sake of simplicity, we derive the formula for the two-dimensional Black and Scholes market. Let $H(T) = \max(S^1(T) - S^2(T), 0)$ be the payoff of an exchange option where S^1 and S^2 denote the two (non-dividend-paying) stocks. There are two sources of uncertainty modeled by two independent standard Brownian motion B^1 and B^2 under the physical measure. Without loss of generality one can assume that under the physical measure,

$$\frac{dS^1(t)}{S^1(t)} = \mu_1 dt + \sigma_1 dB^1(t),$$

$$\frac{dS^2(t)}{S^2(t)} = \mu_2 dt + \sigma_2 dB^2(t).$$

We also suppose that the stocks do not pay dividends. Using Margrabe's formula (1978) for the exchange option in the Black and Scholes framework, it is easy to prove that

$$H^*(T) = \mathrm{E}[H(T) | S_T^{\pi^*}(T)] = S^1(0)\Phi(d^+) - S^2(0)\Phi(d^-),$$

where $d^+ = [\ln(S_1(0)/S_2(0)) + \mathbb{V}T/2]/\mathbb{V}\sqrt{T}$, $d^- = [\ln(S_1(0)/S_2(0)) - \mathbb{V}T/2]/\mathbb{V}\sqrt{T}$, and where

$$\mathbb{V} = \mathbb{V}_1 + \mathbb{V}_2 + 2\sqrt{\mathbb{V}_1\mathbb{V}_2}\rho_{cond}$$

with $\rho_{cond} = [\mathbb{C}_{12} - \mathbb{M}_1\mathbb{M}_2]/\sqrt{\mathbb{V}_1\mathbb{V}_2}$. The conditional moments $\mathbb{M}_1$, $\mathbb{V}_1$, $\mathbb{M}_2$, $\mathbb{V}_2$ are given by (41) and $\mathbb{C}_{12} = E[X^1(T)X^2(T)|X^{\pi^*}(T)]$.

No explicit expression can be obtained for the cdf of $H(T)$, and therefore for $Z^*(T)$. Note, however, that Monte Carlo simulations could be done to compute the payoff $Z^*(T)$.

4.3 Constant Proportion Portfolio Insurance

4.3.1 Model Setup

When pursuing a CPPI strategy the investor first determines floor values F_t for his or her portfolio at future times t, where F_t can be seen as the lowest acceptable value of the portfolio at time t. Here we will assume that F_t evolves according to a fixed deterministic rate $c > 0$:

$$F_t = Fe^{ct}. \tag{53}$$

Next, at each time t one computes the cushion C_t as the excess of the actual portfolio value, denoted by V_t, over the floor value F_t. The amount allocated to one or more risky assets is called the exposure E_t and is given by the following relation:

$$E_t = \begin{cases} lC_t & \text{if } C_t > 0 \\ 0 & \text{if } C_t \leq 0 \end{cases}, \tag{54}$$

where $l > 0$ is reflecting the degree of leverage. The remaining proportion $V_t - E_t$ will then be invested in the risk-free account, which is assumed to grow at a fixed rate $r > 0$.

It is well known that in a one-dimensional Black-Scholes market with continuous trading and without borrowing restrictions a single-asset CPPI gives rise to a path-independent payoff increasing in the underlying asset and thus is cost efficient. However, in reality trading occurs at discrete times and may be subject to borrowing constraints. In this example we will first analyze the performance of a single-asset CPPI; see also Vanduffel et al. (2009). Next, we will extend this performance analysis to include a multiasset CPPI structure.

4.3.2 A Single-Asset CPPI

In this section we analyze a one-dimensional Black-Scholes market where a single-asset CPPI applied on an underlying risky asset with price process $\{S^1(t), t \geq 0\}$ is available. Furthermore the investor is trading at discrete times without transactions costs and is not allowed to go short. In particular this means that the exposure E_t will be capped by V_t from above. We will also assume that the rebalancing occurs on a monthly basis and that there are no transaction costs involved.

The CPPI strategy gives rise to a payoff $H(T)$ at time T, and we apply Monte Carlo methods to estimate the distribution function F of $H(T)$ in a straightforward way. From Theorem 1 we find that the cost-efficient payoff $Z^*(T)$ will be given by

$$Z^*(T) = F^{-1}(F_{S^1(T)}(S^1(T))). \tag{55}$$

Finally, we apply Monte Carlo simulation under the risk-neutral measure $\mathbb{Q}$ to derive the approximate price for $Z^*(T)$, and this will give us an insight into the inefficiency cost of the CPPI strategy.

In Table 1 we show the increases in cost (expressed as percentages) for the CPPI payoffs $H(T)$ as compared to the modified payoff $Z^*(T)$. The results are obtained by running 1,000,000 Monte Carlo simulations. We fixed the parameters $c = r = 0.04$ and also assumed that $\mu_1 = 0.10$. From Table 1 we observe that the inefficiency costs become more pronounced when increasing the standard deviation σ_1 for the (log) return of the underlying risky asset, the time horizon T, or the multiplier l. Intuitively this all makes sense because by doing so the initial payoff $H(T)$ becomes more and more "chaotic" and path dependent, meaning that the realizations v_i for $H(T)$ show less tendency to be increasing in the corresponding realizations s_i for $S^1(T)$, which suggests that the inefficiency is increasing in these instances.

Table 1

Inefficiency Cost of Single-Asset CPPI Structure $\mu_1 = 0.10$, $c = r = 0.04$

$T = 1$					
l/σ_1	0.15	0.20	0.25	0.30	0.35
4	0.0121	0.0205	0.0382	0.0653	0.1082
7	0.0572	0.1225	0.2152	0.3181	0.4229
$T = 3$					
l/σ_1	0.15	0.20	0.25	0.30	0.35
4	0.0803	0.1268	0.2122	0.3187	0.4876
7	0.3529	0.6522	0.9552	1.2169	1.4107
$T = 5$					
l/σ_1	0.15	0.20	0.25	0.30	0.35
4	0.2195	0.3473	0.5233	0.7490	1.0179
7	0.7095	1.2112	1.6502	1.9881	2.2212

4.3.3 A Multiasset CPPI

Here we consider a so-called hybrid CPPI strategy, which involves an exposure to m different underlying single-asset CPPIs, each of them written on a risky asset with price process $\{S^i(t), t \geq 0\}$, $i = 1, \ldots, m$. As before there are no transaction costs, and short selling will not be allowed.

The strategy we consider effectively consists in purchasing at $t = 0$ a portfolio of single-asset CPPIs. Next, as time evolves one aims at keeping the initial proportions invested in each of the underlying CPPIs constant. Hence a periodical rebalancing of the portfolio to maintain the initial weights invested in the underlying CPPIs will be required; that is, at the end of each period the investment in the better performing CPPI will be reduced in favor of the investment in the less performing one. As compared to a standard single-asset CPPI the perceived advantages of such multidimensional CPPI structure are a lower volatility on the investment return (because of the diversification) and a lower exposure to gap risk. The cost-efficient payoff $Z^*(T)$ is given by

$$Z^*(T) = F^{-1}(F_{S^{\pi^*}(T)}(S^{\pi^*}(T))), \tag{56}$$

where F^{-1} is a quantile function corresponding to the payoff $H(T)$ of the CPPI strategy. As before Monte Carlo simulation under the risk-neutral measure $\mathbb{Q}$ is then used to estimate the initial price of the new structure $Z^*(T)$.

For the numerical examples in this section, we choose to analyze a hybrid CPPI, which consists of pursuing a periodical rebalancing investment strategy in two underlying plain-vanilla CPPIs; that is, we have that $m = 2$. Within each CPPI the reallocation occurs on a monthly basis, whereas for the rebalancing across the CPPIs a six-monthly reallocation scheme has been applied.

We ran 1,000,000 Monte Carlo simulations assessing different values for the time horizon T, the multiplier l, and the initial weights invested in each CPPI. The drift parameters have been determined as $\mu_1 = 0.08$ and $\mu_2 = 0.12$, whereas for the volatilities we have chosen $\sigma_1 = 0.20$ and $\sigma_2 = 0.30$. For the correlation between the risky log-returns we take $\rho = 0.25$. Finally we have assumed that $c = r = 0.04$. In Table 2 we show the increases in cost (expressed as percentages) for the hybrid CPPI payoff $H(T)$ as compared to the modified payoff $Z^*(T)$. As documented previously in the one-dimensional example, the inefficiency costs become more pronounced when increasing the time horizon T or the multiplier l. As for the impact on the results when varying the constant exposures to the underlying single-asset CPPIs, we observe that the cost inefficiency is more substantial in the case where most of the initial wealth has been invested in one of both single-asset CPPIs. Conversely, when the portfolio insurance weights are more "balanced," we find that the strategies become more optimal in terms of cost inefficiency but still remain inferior to the path-independent alternatives.

Table 2
Ineffciency Cost of Multiasset CPPI Structure
$c = r = 0.04$, $\mu = (0.08, 0.12)$, $\sigma = (0.2, 0.3)$, $\rho = 0.25$

$T = 1$					
l/weights	[0.1 0.9]	[0.3 0.7]	[0.5 0.5]	[0.7 0.3]	[0.9 0.1]
4	0.0905	0.0431	0.0284	0.0254	0.1191
7	0.3215	0.2082	0.1417	0.1452	0.2404
$T = 3$					
l/weights	[0.1 0.9]	[0.3 0.7]	[0.5 0.5]	[0.7 0.3]	[0.9 0.1]
4	0.3978	0.2360	0.1290	0.1619	0.3921
7	1.1905	0.7824	0.5626	0.5864	0.8375
$T = 5$					
l/weights	[0.1 0.9]	[0.3 0.7]	[0.5 0.5]	[0.7 0.3]	[0.9 0.1]
4	0.7952	0.4921	0.3234	0.3891	0.6583
7	2.0567	1.4078	1.0356	1.0325	1.4904

5. CONCLUSIONS

We have examined the optimality of payoffs in multidimensional Black-Scholes markets from the point of view of profit-seeking consumers with a fixed investment horizon and who have law-invariant preferences (including the expected utility framework as a special case). In particular we have also assumed that consumption will happen only at the end of the horizon and not at intermediate times. Under these assumptions we have shown that path-dependent payoffs are not efficient and that optimal payoffs can be expressed as path-independent derivatives written on the underlying market portfolio.

Our findings cast doubt on the performance of complex (path-dependent) financial payoffs and motivate simpler (path-independent) designs for these. However, although the existence of path-independent payoffs cannot be really justified in the classical utility theory (as well as any law-invariant framework such as, for instance, Yaari's dual theory, the cumulative prospect theory, or the rank-dependent utility theory), it maybe explainable assuming other criteria for decision making. For example, Boyle and Tian (2008) consider an investor who aims at outperforming a (stochastic) benchmark with sufficiently large probability. They show that the optimal payoff for a utility maximizer is not necessarily the optimal contract in this context. In this study, when the benchmark is random, the preferences are not law invariant but become state dependent. The results discussed in our paper do not hold.

APPENDIX

PROOFS

A.1 Proof of Proposition 2

PROOF

We have that

$$
\begin{aligned}
E[\xi_t S_t^i] &= S_0^i E[e^{-rt} \cdot e^{-\theta^2/2t - \theta B^{\pi^*}(t)} \cdot e^{(\mu_i - 1/2\sigma_i^2)t + \sigma_i B^i(t)}] \\
&= S_0^i e^{-rt} \cdot e^{-\theta^2/2t} e^{(\mu_i - 1/2\sigma_i^2)t} \cdot E[e^{-\theta B^{\pi^*}(t) + \sigma_i B^i(t)}] \\
&= S_0^i e^{-rt} \cdot e^{-\theta^2/2t} e^{(\mu_i - 1/2\sigma_i^2)t} \cdot e^{1/2 Var[-\theta B^{\pi^*}(t) + \sigma_i B^i(t)]} \\
&= S_0^i \cdot e^{-rt} \cdot e^{-\theta^2/2t} \cdot e^{(\mu_i - 1/2\sigma_i^2)t} \cdot e^{\theta^2/2t} e^{\sigma_i^2/2t} e^{-\theta\sigma_i \cdot Cov[B^{\pi^*}(t), B^i(t)]} \\
&= S_0^i \cdot e^{-rt} \cdot e^{\mu_i t} \cdot e^{-\theta/\sigma(\pi^*) Cov[\sigma(\pi^*) B^{\pi^*}(t), \sigma_i B^i(t)]}.
\end{aligned} \tag{A.1}
$$

Further from (15) and(16) we find that

$$\frac{\theta}{\sigma(\pi^*)} = \mathbf{1}^T \cdot \Sigma^{-1} \cdot (\mu - r\mathbf{1}), \tag{A.2}$$

and hence using (17) we find that

$$\begin{aligned} E[\xi_t S_t^i] &= S_0^i \cdot e^{-rt} \cdot e^{\mu_i t} \cdot e^{-(\mu_i - r)t} \\ &= S_0^i. \end{aligned} \tag{A.3}$$

This ends the proof. ■

A.2 Proof of Theorem 1

PROOF

We now recall Theorem 1 of Bernard and Boyle (2010) that holds in a multidimensional market, and that gives the explicit representation for cost-efficient strategies.

Theorem 4 (Bernard and Boyle 2010)

Denote by F_ξ the cdf of ξ_T. Let F be a cdf. Define

$$Y_T^* = F^{-1}(1 - F_\xi(\xi_T)). \tag{A.4}$$

Then Y_T^* is distributed with the cdf F. For any other random variable Y with the same cdf F, $c(Y_T^*) \leq c(Y)$, which means Y_T^* minimizes the cost of achieving a payoff with distribution F. If in addition $c(Y_T^*) < +\infty$, then Y_T^* is the almost surely unique optimal solution to achieve the cheapest payoff with cumulative distribution function F.

This theorem holds because in a multidimensional Black-Scholes market, the state price process is obviously continuously distributed, which is the key assumption in the derivations of Bernard and Boyle (2010). In their paper they also discuss the strong connections between cost efficiency and stochastic dominance, a connection first pointed out by Dybvig (1988).

Theorem 5 (Dybvig 1988)

Consider a payoff $H(T)$ with cdf F. Taking into account the initial cost of the payoff, the cost-efficient payoff Y_T^* of the payoff $H(T)$ is a.s. equal to $H(T)$ or dominates $H(T)$ in the first-order stochastic dominance sense. The dominance ordering is strict unless $H(T)$ is a nonincreasing function of ξ_T.

The result of Theorem 1 then holds because $F_{S^{\pi^*}(T)}(S^{\pi^*}(T))$ is the same variable as $1 - F_\xi(\xi_T)$. Indeed, they are two comonotonic random variables with a uniform distribution. They have to be equal almost surely. The comonotonicity comes from the relationship (23) that exists between $S^{\pi^*}(T)$ and ξ_T. ■

A.3 Proof of Theorem 3

PROOF

We are confronted with the following optimization problem:

$$\max_{H(T)} \mathrm{E}_{\mathbb{P}}[u(H(T))] \text{ s.t. } \quad W_0 = \mathrm{E}_{\mathbb{P}}[\xi(T) \cdot H(T)]. \tag{A.5}$$

The solution for such an optimization problem (A.5) is well known (see, for instance, Korn 1997, Chapter 3, Section 4), and one obtains that the unique optimal payoff $H^*(T)$ for problem (A.5) is equal to

$$H^*(T) = (u')^{-1}(\lambda\xi(T)), \tag{A.6}$$

with λ determined by

$$W_0 = \mathrm{E}_{\mathbb{P}}[\xi(T)H^*(T)]. \tag{A.7}$$

Next, we observe that $\xi(T)$ can be written as an explicit function of the terminal value $S^{\pi^*}(T)$ of the market portfolio π^* as given by (23) in Proposition 2. Hence we find that $H^*(T)$ can also be expressed as

$$H^*(T) = h^*(S^{\pi^*}(T)), \tag{A.8}$$

with $x \rightarrow h^*(x)$ given by

$$h^*(x) = (u')^{-1}(cx^{-\theta_*/\sigma_*}), \tag{A.9}$$

where $c > 0$ is such that

$$W_0 = E_{\mathbb{P}}[\xi(T)h^*(S^{\pi^*}(T))]. \tag{A.10}$$

This ends the proof. ∎

A.4 Proof of Equation (37) in Example 1

PROOF

To find the exact expression of $H^*(T)$, one needs to verify that it satisfies (36) (this problem can also be solved directly by looking for the constant c involved in the expression (35) of $H^*(T)$).

This is not difficult because

$$E_{\mathbb{P}}[\xi(T)H^*(T)] \tag{A.11}$$

is the expectation of a lognormal distribution. We need only the mean and the variance of this lognormal distribution. Observe that

$$\xi(T) = e^{-rT-\theta_*^2T/2-\theta_*B^*(T)},\ S^*(T) = S^*(0)e^{(\mu_*-\sigma_*^2/2)T+\sigma_*B^*(T)}. \tag{A.12}$$

Therefore after some straightforward computations

$$E_{\mathbb{P}}[\xi(T)H^*(T)] = W_0 \tag{A.13}$$

implies that

$$H^*(T) = W_0 \cdot e^{(1-\alpha)rT+\alpha X^{\pi^*}(T)+(\alpha-\alpha^2)\sigma_*^2/2T}, \tag{A.14}$$

with α given by

$$\alpha = \frac{1}{\gamma}\frac{\theta_*}{\sigma_*}. \tag{A.15}$$

Further, using (11) and(9) we find that $H^*(T)$ can also be expressed as

$$H^*(T) = W_0 \cdot e^{((1-\alpha)r+\alpha\mu_*-\alpha^2\sigma_*^2/2)T+\alpha\sigma_*B^*(T)}. \tag{A.16}$$

On the other hand, using (11) and taking into account (8) we find that

$$X^{\pi_\alpha^*}(T) = \left((1 - \alpha)r + \alpha\mu_* - \frac{\alpha^2\sigma_*^2}{2}\right) T + \alpha\sigma_*B^*(T). \tag{A.17}$$

This ends the proof. ∎

A.5 Proof of Equation (52)

PROOF

Define $F^*(x) = P(H^*(T) \leq x)$. For $x < W_0(1-\alpha)e^{rT}$, then

$$F^*(x) = \Phi\left(\frac{\frac{1}{c_1}\ln\left(\frac{\alpha W_0 e^{b_1}}{x - W_0(1-\alpha)e^{rT}}\right) + \left(\mu_* - \frac{\sigma_*^2}{2}\right)T}{\sigma_*\sqrt{T}}\right).$$

For $x \geq W_0(1-\alpha)e^{rT}$, one has $F^*(x) = 1$.

One can then calculate the inverse $(F^*)^{-1}$:

$$(F^*)^{-1}(y) = \begin{cases} W_0(1-\alpha)e^{rT} & \text{if } y = 1 \\ \alpha W_0 e^{b_1}\exp\left(c_1\left(\mu_* - \frac{\sigma_*^2}{2}\right)T - c_1\sigma_*\sqrt{T}\Phi^{-1}(y)\right) + (1-\alpha)W_0 e^{rT} & \text{otherwise.} \end{cases}$$

Then $G^*(T) = (F^*)^{-1}(F_{S^{\pi^*}}(S^{\pi^*}(T)))$, and formula (52) is straightforward to obtain. ■

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