Protection of a Company Issuing a Certain Class of Participating Policies in a Complete Market Framework

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Abstract

In this article we examine to what extent policyholders buying a certain class of participating contracts (in which they are entitled to receive dividends from the insurer) can be described as standard bondholders. Our analysis extends the ideas of Bühlmann and sequences the fundamental advances of Merton, Longstaff and Schwartz, and Briys and de Varenne. In particular, we develop a setup where these participating policies are comparable to hybrid bonds but not to standard risky bonds (as done in most papers dealing with the pricing of participating contracts). In this mixed framework, policyholders are only partly protected against default consequences. Continuous and discrete protections are also studied in an early default Black and Cox–type setting. A comparative analysis of the impact of various protection schemes on ruin probabilities and severities of a life insurance company that sells only this class of contracts concludes this work.

1. Introduction

The last two decades have seen the emergence of an increasing number of papers bridging the conceptual and practical void between financial and actuarial theories. During the last decade, practitioners and academics have called for greater standardization. The new regulatory environment, strongly inspired by Anglo-American practices, has also called for further development of market-based pricing tools. See in particular Ballotta, Esposito, and Haberman (2006) for a detailed account on the enforcement and implications of the new IAS/IFRS/Solvency II norms and Bühlmann (2002, 2004) for an insight into market valuation. The present study is devoted to the calculation of appropriate premia and safety loadings for a certain class of participating contracts (mortality risk is not considered or supposed to be fully diversified). The policies under study are contracts in which policyholders and equityholders share the benefits of the insurance company. Thus, policyholders receive some additional payments, which will be referred to as dividends. They are also called ''bonuses'' or ''participation rate'' in European countries.

The framework adopted here dates back to the analysis of the corporation initiated by Merton (1974). The essence of this approach is understanding equity as a call option on the firm's assets, and risky debt as the sum of risk-free debt plus a short position on a ''default'' put on the assets. This approach is also the one chosen by Briys and de Varenne in their papers on life insurance (see their book [2001] for a general treatment). It has been extended, under an assumption of stochastic interest rates, by Bernard et al. (2005) in the wider Black and Cox (1976) context that enlarges Merton's framework by

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considering that bankruptcy is possible at any moment. This study also builds on the framework of Bühlmann (2002, 2004), where the relevance of replication arguments is highlighted.

Among the related literature, we can cite Schweizer (2001), who proposes a financial valuation principle that is derived from traditional actuarial premium calculations, but at the same time takes into consideration the possibility of trading in a financial market. In a similar vein, Boyle and Tian (2008) take into account the profit margin and the safety loading in the pricing of equity-indexed annuities. These contracts are very similar to the policies we are studying. There is a minimum guaranteed rate and a participating coefficient. These authors found that the premium paid by an investor is never equal to the market value of the contract, because of the safety loading and profit margin of the company. They propose to use for the pricing of such contracts a minimum guaranteed rate and a participation coefficient lower than in a fair contract.

In this article we question the idea that policyholders are short of a default put on the insurer's assets. In other words, we examine to what extent policyholders can be described as standard bondholders, according to the lines of Merton and followers. Indeed, it appears doubtful that this particular class of participating policies can simply be priced in terms of exotic bonds. In particular, we develop a setup in which life insurance policies are comparable to hybrid bonds but not to standard risky bonds (as done in most papers dealing with the pricing of these contracts). In this mixed framework, policyholders are only partly protected against default consequences.

Thus, we consider life insurance contracts as hybrid debt where the importance of the security loading is related to the importance of the debt/equity nature of contracts. The first section introduces this new mixed framework. The second section is dedicated to its extension to the case when bankruptcy can happen at any time. Continuous and discrete protection schemes are studied in this setting. The third section proposes a comparison of three distinct types of protection: the mixed protection introduced in this article, a protection consisting of a simple increase of the assets backing the liabilities, and a protection made of equity default swaps. In particular, the impact of these protections on ruin probabilities and severities is analyzed.

2. TOWARD A UNIFIED FRAMEWORK

We start by briefly reviewing the standard application of the financial theory of the firm (as first developed by Merton 1974) in life insurance. Then we question the direct application of this theory in life insurance and propose a new paradigm in which safety loadings play a central role.

2.1 Use of Financial Theory in Life Insurance

The first papers applying the contributions of Black, Scholes, and Merton in a life insurance context are those of Boyle and Schwartz (1977) and Brennan and Schwartz (1976). These authors value simple guarantees as options, under a constant interest rate structure. Since these papers, various other papers have appeared, including the important contributions of Briys and de Varenne (1994), who more fully develop the pricing of these contracts.

The fundamental idea underlying the above-mentioned literature is that Merton's capital structure of the corporation can be directly translated in insurance. This yields the balance sheet in Table 1,

Table 1
Initial Capital Structure of a Simple Life Office

Assets	Liabilities
A_0	E_0 L_0

where liabilities are composed of the initial equity E_0 and of the initial contribution L_0 by policyholders. E_0 and L_0 together are invested in the assets A_0.

The current literature assumes that policyholders (as opposed to stockholders) face full liability with respect to a possible bankruptcy. Thus, it assumes that policyholders are identical to bondholders and that these insurance policies can be valued using the standard financial approach. As far as participating contracts are concerned, their pricing then simply boils down to the pricing of particular exotic contracts.

A key assumption is that the company is issuing only one type of contracts. Because contracts are all identical, the possible default of the company is the same as the default on the contracts under study.

We denote by α the proportion of assets initially owned by policyholders ($\alpha = L_0/A_0$). Let r_g be the minimum guaranteed rate (fixed at time 0). Consider, for instance, a participating contract guaranteeing at maturity $L_T^g := L_0 e^{r_g T}$ and an exogenous dividend rate δ (also called "participation rate"). Its payoff can be expressed as

$$\Theta_L(T) = \begin{cases} A_T & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \leq A_T \leq \dfrac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \dfrac{L_T^g}{\alpha}. \end{cases} \tag{2.1}$$

In the first state, bankruptcy is declared, and policyholders recover the residual asset value, while in the second state only the guaranteed amount is distributed. In the third, beneficial, situation, a dividend is offered in addition to the guaranteed amount.

This payoff at time T admits the compact form

$$X(g, \delta, T) := L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+. \tag{2.2}$$

From this expression, one readily understands what a participating contract is (within the standard paradigm): *a guaranteed amount, plus a long position in a call on the assets, plus a short position in a put on the identical assets.* The call option corresponds to the dividend. The (short) put option corresponds to a default put, as defined in financial markets.

In an arbitrage-free complete market, the value V_0 of the contract can be obtained directly under the unique risk-neutral measure Q of this market as

$$V_0 = \mathbb{E}_Q\left[e^{-\int_0^T r_s\,ds}\left(L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+\right)\right], \tag{2.3}$$

where r_s is the instantaneous risk-free rate at time s. Markets are supposed to be sufficiently liquid and integrated for this to hold.

When this type of contract is sold by a life insurance company, it usually involves mortality risk. The maturity date can then be conditional on the policyholder being still alive at time T, and the payoff (2.2) can be a death benefit and therefore paid at time of death instead of at a fixed maturity T. In the remainder of this article, we ignore mortality risk. However, the present study can be readily extended to include mortality risk when it is fully diversified.

Let us illustrate with an example how this can be done. Assume that the payoff (2.2) is paid at the end of the year of death if it occurs before a fixed date T or at time T if the policyholder is still alive at time T: the maturity of the contract is now random. The probability distribution of the time of death is given in life tables. Let us denote by q_{x+t} the probability that a policyholder of age $x + t$ will die before the end of the year, with $t = 0, \ldots, T - 1$, and let $p_{x+t} = 1 - q_{x+t}$. If we assume that the mortality is fully diversified, then the actual deaths corresponding to a large number N of identical policies issued by the company follow exactly the life tables. The insurer can therefore anticipate the approximate number of contracts that will terminate every year. Mortality risk is then hedged by pooling risks, that is, by selling as many identical policies as possible. Thus the contract's market value at time 0 for an individual of age x, denoted V_0, is simply given by

$$V_0 = {}_Tp_x \cdot \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}X(g,\ \delta,\ T)] + \sum_{t=0}^{T-1} {}_tp_x \cdot q_{x+t} \cdot \mathbb{E}_Q[e^{-\int_0^{t+1} r_s\,ds}X(g,\ \delta,\ t+1)], \tag{2.4}$$

where X is defined above by formula (2.2) and ${}_tp_x = \Pi_{j=0}^{t-1}\ p_{x+j}$ is the probability that an individual of age x survives at least t years. It is the same idea as in Bernard and Lemieux (2008), who show how to incorporate mortality risk into the pricing of life insurance contracts with index-linked minimum guaranteed death benefits without explicitly simulating mortality.

Observe that the market value of the contract given by formula (2.4) is now a linear combination of contracts with *fixed* maturities. The insurer collects a large number of premiums, say, NV_0. There are approximately N_Tp_x contracts that will mature at time T, and $N_tp_x \cdot q_{x+t}$ that will terminate at time $t+1$ for any time $t \in \{0, \ldots, T-1\}$.

For the sake of simplicity, we study a contract with fixed maturity in the remainder of the paper. Using expression (2.4), it is straightforward to extend our study to the general case when mortality risk is taken into account but fully diversified.

2.2 Market Model

Throughout the paper the company's assets A follow a geometric Brownian motion, and we use a particular Hull and White model for interest rates. The volatility structure is thus exponential. Given $\nu > 0$ and $a > 0$, it can be written as

$$\sigma_P(t,\ T) = \frac{\nu}{\alpha}(1 - e^{-a(T-t)}). \tag{2.5}$$

Under the risk-neutral probability measure Q, the assets value A and the zero-coupon bond price with expiry date T, $P(., T)$, follow the classical SDEs:

$$\begin{cases} \dfrac{dA_t}{A_t} & = r_t\,dt + \sigma\,dZ^{Q}(t) \\ \dfrac{dP(t,T)}{P(t,T)} & = r_t\,dt - \sigma_P(t,T)\,dZ_1^{Q}(t), \end{cases} \tag{2.6}$$

where $Z^Q(.)$ and $Z_1^Q(.)$ are Q-standard correlated Brownian motions (ρ is their correlation coefficient). See, for instance, Bernard et al. (2005), page 503, section 1.5, for more details on this framework.

In this setting, it is easy to compute this value in closed form, even with stochastic interest rates, when it is assumed that default can occur only at maturity (see, e.g., Bernard et al. 2006, the computation of [2.4], p. 182). To eliminate the stochastic discount factor, we move to the T-forward neutral universe. V_0 can then be expressed as

$$V_0 = P(0,T)\,[L_T^g + \delta\alpha E_1 - \delta L_T^g E_2 - L_T^g E_3 + E_4], \tag{2.7}$$

where

$$E_1 = \mathbb{E}_{Q_T}[A_T \mathbf{1}_{A_T > (L_T^g/\alpha)}], \qquad E_3 = Q_T[A_T < L_T^g],$$

$$E_2 = Q_T\left[A_T > \frac{L_T^g}{\alpha}\right], \qquad E_4 = \mathbb{E}_{Q_T}[A_T \mathbf{1}_{A_T < L_T^g}],$$

where Q_T denotes the forward-neutral probability, and where E_1, E_2, E_3, and E_4 can be computed in closed form under the hypotheses given above:

$$E_1 = \Phi_1\left(\mathrm{M}_T;\ \sqrt{\mathrm{V}_T};\ \frac{L_T^g}{\alpha}\right), \qquad E_3 = \mathrm{N}\left(\frac{\ln(\mathrm{L}_\mathrm{T}^g) - \mathrm{M}_\mathrm{T}}{\sqrt{\mathrm{V}_\mathrm{T}}}\right),$$

$$E_2 = \mathrm{N}\left(\frac{\mathrm{M}_\mathrm{T} - \ln(\mathrm{L}_\mathrm{T}^g/\alpha)}{\sqrt{\mathrm{V}_\mathrm{T}}}\right), \qquad E_4 = \Phi_2(\mathrm{M}_T;\ \sqrt{\mathrm{V}_T};\ L_T^g),$$

where M_T and V_T are the two moments of the $\ln(A_T e^{-r_g T})$:

$$\begin{cases} \mathrm{M}_T = \ln\left(\dfrac{A_0}{P(0,T)}\right) + \dfrac{\nu^2}{4a^3} - \left(\dfrac{\nu^2}{2a^2} + \dfrac{\rho\sigma\nu}{\alpha} + \dfrac{\sigma^2}{2} + r_g\right) T - \dfrac{\nu^2}{4a^3}\, e^{-2aT} \\ \qquad \times \left(\dfrac{\nu^2}{2a^3} + \dfrac{\rho\sigma\nu}{a^2}\right) - \left(\dfrac{\nu^2}{a^3} + \dfrac{\rho\sigma\nu}{a^2}\right) e^{-aT} + \dfrac{\nu^2}{2a^3}\, e^{-2aT}, \\ \mathrm{V}_T = 2\nu\, \dfrac{\nu + \alpha\rho\sigma}{a^3}\, e^{-aT} - \dfrac{\nu^2}{2a^3}\, e^{-2aT} - \dfrac{3\nu^2}{2a^3} - \dfrac{2\rho\sigma\nu}{a^2} + \left(\sigma^2 + \dfrac{2\rho\sigma\nu}{a} + \dfrac{\nu^2}{a^2}\right) T, \end{cases}$$

where N is the c.d.f. of the centered reduced normal distribution, and where Φ_1 and Φ_2 are defined by

$$\begin{cases} \Phi_1(m;\sigma;a) = \mathbb{E}[e^X \mathbf{1}_{e^X > a}]\ \exp\left(m + \dfrac{\sigma^2}{2}\right) \mathrm{N}\left(\dfrac{m + \sigma^2 - \ln(a)}{\sigma}\right) \\ \Phi_2(m;\sigma;a) = \mathbb{E}[e^X \mathbf{1}_{e^X < a}] = \exp\left(m + \dfrac{\sigma^2}{2}\right) \mathrm{N}\left(\dfrac{\ln(a) - m - \sigma^2}{\sigma}\right) \end{cases} \tag{2.8}$$

with X a normal random variable following the $\mathcal{N}(m, \sigma^2)$ distribution.

The contract considered here is priced, according to the literature, as a type of defaultable bond. However, policyholders may not want to take the position of bondholders (to be short the whole default put) and may require that the company protects itself from default.

We show throughout the remainder of this paper that policyholders' and bondholders' positions may actually differ. In this respect, safety loadings are of utmost importance to achieving a better understanding of this problem.

2.3 A First Adaptation of the Paradigm

Our concerns can be expressed simply as follows:

Bondholders know they are betting on the insolvency probability of the firm. They expect additional return to compensate for these risks. Policyholders (especially for long-term life insurance) aim at purchasing default-free entities. Life companies thereby impose safety loadings on insurance premia to compensate for bankruptcy potential.

A simplified answer for a life insurance company could be to sell back the whole default put to policyholders. In this approach the payoff to policyholders is always positive and no bankruptcy is possible (in particular because the company charges much more to policyholders at issuance, and because of replication arguments). This additional charge can be interpreted as the safety loading.

In the case of a participating contract, we obtain the payoff:

$$\hat{\Theta}_L(T) = \begin{cases} L_T^g & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \le A_T \le \dfrac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \dfrac{L_T^g}{\alpha}, \end{cases} \tag{2.9}$$

Table 2
Model Parameters

A_0	σ	T	α	a	ν	$P(0, T)$	ρ	δ	r_g
100	10%	10	0.9	0.4	0.007	0.6703	−0.05	91.68%	2%

where it can be seen that policyholders are, in all circumstances, truly guaranteed the amount L_T^g. This payoff can be written in compact form as follows:

$$L_T^g + \delta(\alpha A_T - L_T^g)^+, \tag{2.10}$$

which amounts to a guarantee, plus a simple call option.

Hence, the market value of the secured contract becomes

$$\hat{V}_0 = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\ (L_T^g + \delta(\alpha A_T - L_T^g)^+)].$$

The secured contract's premium $\hat{V}_0$ is paid at time 0 and is higher than the risky contract's premium V_0 considered before. Note that the initial value S_0 of the safety loading (equal to the difference between the value of the secured contract $\hat{V}_0$ and the risky one V_0) is matched exactly by the initial price of the Merton default put. In particular,

$$S_0 = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\ (L_T^g - A_T)^+],$$

where one readily has $\hat{V}_0 = V_0 + S_0$. We assume that V_0, together with the initial investment of equityholders, is used to constitute the assets of the fund ($A_0 = V_0 + E_0$), and that S_0 is used to buy a product yielding the payoff $(L_T^g - A_T)^+$ at time T on the market. If this put on the assets can be found or duplicated in the market, the contract becomes risk free (its payoff is given by [2.9]) and the probability of bankruptcy nil. Another possibility is to invest $\hat{V}_0 = V_0 + S_0$ in the global fund along with the shareholders's initial investment. In the absence of an investment strategy, the default probability is reduced but still positive (see Ballotta, Esposito, and Haberman 2006).

Let us illustrate the previous discussion with a short numerical example. We specify our model parameters in Table 2.

The assets' volatility σ is set at 10%, which corresponds to a standard investment (approximately half in stocks and half in bonds). We assume that the contract maturity T is equal to 10 years, and α is the initial participation of the insured in the capital structure of the firm. The parameters a, ν define the zero-coupon volatility, while ρ is the correlation coefficient between the asset-generating process and the instantaneous interest rate process. Finally, r_g is the minimum guaranteed rate and δ is the participating coefficient: they are such that the risky contract sold to policyholders is fair. So we set $V_0 = \alpha A_0 = L_0 = 90$. Table 3 displays the participating contract values computed using the parameters defined in Table 2.

At first sight, the initial premia of the two contracts V_0 and $\hat{V}_0$ are close (with $\hat{V}_0 = V_0 + S_0$). However, one can observe that $\frac{S_0}{V_0}$ is approximately worth 2.7%. A simple approximation would yield an impact of 0.27% in terms of annual return (due to the 10Y maturity of the product), which is compared against

Table 3
Results

V_0	$\hat{V}_0$	S_0
90	92.42	2.42

a 2% annual guaranteed rate. Indeed, making the company (or a contract) safe is costly, and making it utterly safe is even more so. Of course, a higher σ would entail a higher discrepancy between V_0 and $\hat{V}_0$.

The contract is here fully protected, but the price of perfect coverage is relatively high (having the effect of reducing the appeal of such a contract). Below we consider a mixed situation where opportunity is introduced for smaller safety loadings.

2.4 A Mixed Framework

It seems unlikely that the aforementioned safety loading can be fully charged. Risk-return considerations are just as important for people buying participating policies. We claim that policyholders buy policies that are more or less protected, depending on their risk and return preferences. On the other hand, life offices will guarantee the insured's amount fully, or partly, depending on how much security loading they may levy.

Therefore, our proposed solution is that life insurance companies sell back a portion of the default put to policyholders, but a portion only. The higher this portion of the default put is sold back, the higher the corresponding security loading is. To make this even more explicit, we construct a simple linear model of default puts/security loadings where a protection coefficient ψ is introduced. PSI stands for *p*olicyholder's *i*mmunization coefficient. When ψ is equal to zero, the default put is not sold back to policyholders; they remain entirely short of the default put. This is simply the implicit assumption as taken from the existing literature. When ψ is equal to one, the security loading is complete, and the whole default put is consequently sold back to policyholders. In this situation the contract offers a much lower return than under the preceding situation (i.e., the contract is very secure, but very expensive). Our opinion is that the factor ψ has to be strictly bounded between 0 and 1 to model adequately existing insurance practices.

Thus, we introduce the parameter ψ that describes the amount of security loading charged by a life insurance company, and we observe that it is proportional to the amount of default put sold back to policyholders.

We describe a general linear framework where, upon bankruptcy, policyholders do not recover the entire "guaranteed amount," but are not completely penalized either by the inferior performance of the assets. We give the following payoffs in the mixed approach:

$$\hat{\hat{\Theta}}_L(T) = \begin{cases} \psi\, L_T^g + (1-\psi)\, A_T & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \le A_T \le \dfrac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \dfrac{L_T^g}{\alpha}. \end{cases} \tag{2.11}$$

In the first situation ($A_T < L_T^g$) a mixed amount of the asset value A_T and of the officially guaranteed amount L_T^g is recovered. This state corresponds to the instance where the company could not avoid default, but could, by an appropriate investment strategy, limit the severity of losses, and distribute back more than A_T.

The above payoff can be written in the following compact form:

$$\hat{\hat{\Theta}}_L(T) = L_T^g + \delta(\alpha A_T - L_T^g)^+ - (1-\psi)(L_T^g - A_T)^+. \tag{2.12}$$

Both payoff expressions are general and return the expressions in formulas (2.1), (2.2), (2.9), and (2.10) by assuming, respectively, $\psi = 0$ and $\psi = 1$. It appears clearly in (2.12) that the proportion ψ of the default put is sold back to policyholders (meaning that this amount of default put is purchased on the open market by the company to protect itself).

The risk-neutral formula for the contract is obtained straightforwardly as

$$\hat{\hat{V}}_0(\psi) = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\,(L_T^g + \delta(\alpha A_T - L_T^g)^+ - (1-\psi)(L_T^g - A_T)^+)],$$

where the total default put is still valued according to

$$S_0 = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\,(L_T^g - A_T)^+],$$

which yields in closed form

$$S_0 = P(0,T)[L_T^g E_3 - E_4].$$

However, the safety loading becomes a fraction ψ of the default put:

$$\hat{\hat{S}}_0(\psi) = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\,\psi(L_T^g - A_T)^+] = \psi S_0, \tag{2.13}$$

and we have the obvious relationship: $\hat{\hat{V}}_0 = V_0 + \hat{\hat{S}}_0$.

Whatever the value of ψ, we are working with a company whose capital structure can be written down as in Table 4, where $\hat{\hat{S}}_0$ is the market value of the safety loading. On the assets side, one can easily conceive that the new line corresponding to $\hat{\hat{S}}_0$ is a derivative position protecting the managed portfolio (corresponding to A_0). On the liability side, the bankruptcy protection is ultimately assigned to policyholders, because it is of no relevance to shareholders.

Let us briefly explain how it is possible to recover the safety loading coefficient ψ of a given company. We omit the different costs related to the marketing of contracts and the management of the company. V_0^m is the price at which a company sells the contract. The market value of a risky contract was previously denoted by V_0. The amount $V_0^m - V_0$ is therefore the amount a policyholder spends in addition to the risky contract: it is the safety loading $\hat{\hat{S}}_0$, which in our framework is equal to ψS_0. Thus the simple formula holds:

$$\psi^m = \frac{V_0^m - V_0}{S_0},$$

where ψ^m is the target safety loading coefficient.

The parameter ψ can be a comparison tool between different lines of business or different contracts. Indeed, the higher ψ is, the more expensive the contract is. Here ψ represents the level of safety loading and at the same time the default risk of the insurer. Customers are willing to buy more expensive contracts if these are safer ones. In the context of property-liability insurance, Sommer (1996) investigates the level of safety loadings using empirical data. He proves that insurance prices reflect the insolvency risk of insurers. This explains in particular why customers are willing to pay different prices for similar contracts. It might thus depend on their personal risk aversion.

We can also interpret ψ as a static risk measure directly constrained by regulators. Higher premia mean more protection is sought. Note that

$$\psi S_0 = \psi \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\,(L_T^g - A_T)^+].$$

In case of default (i.e., $A_T < L_T^g$), the shortfall is $L_T^g - A_T$. Thus ψS_0 is directly linked to the market value of the expected shortfall. This is an important quantity because North American countries have recently adopted the conditional tail expectation as a risk assessment criterion.

Table 4
Initial Capital Structure of a Life Office

Assets	Liabilities
A_0	$E_0 = (1-\alpha)A_0$
$\hat{\hat{S}}_0$	$L_0 = \alpha A_0 + \hat{\hat{S}}_0$

Rating agencies clearly have an important impact on ψ. For example, investors will trust those companies that appear wealthy and will thus agree to pay them higher premia. Thus, similar contracts issued by differently managed companies can be sold at different prices.

Finally, our modeling of safety loadings also reveals the main difference between financial pricing and insurance pricing. In finance, the no-arbitrage principle holds, and prices are uniquely determined and independent of any preferences. In insurance, prices of similar products might differ. A risk-averse insured prefers to buy an expensive policy (a policy issued by a more secure vehicle). Can we consider two products identical, when they are identically denominated but sold by differently rated companies? Our answer is no. There is, in fact, no contradiction between the uniqueness of prices in finance and their apparent multiplicity in insurance. Again, similar products issued by companies protected differently will have different prices. These products although similar cannot be considered identical (credit risk is the main difference between them).

In the following, we extend the protection of life insurance companies to a continuous-time setting, which amounts to assuming a high frequency of regulatory controls taken to the continuous limit.

3. Early Default Setting

To start with, we recall how the existing literature prices unprotected participating contracts when default can happen at any time and interest rates are stochastic. Then we concentrate on two distinct ways of protecting these contracts.

3.1 Unprotected Policies (Early Default Setting)

Let there be, in all situations, a terminal amount $L_T^g = L_0 e^{r_g T}$ guaranteed at maturity T, where r_g is the rate promised to the investors. Note that because of regulatory constraints this rate is often significantly smaller than the rate on treasuries. The default barrier can be defined as the discounted value at r_g of the terminal guaranteed amount:

$$L_t^g = L_T^g e^{-r_g(T-t)}. \tag{3.1}$$

It can also be constructed as follows:

$$L_t^g = L_T^g P(t, T), \tag{3.2}$$

which is the terminal guaranteed amount discounted against a risk-free zero-coupon bond.

Note that the second instance imposes a smaller default barrier than the first one. This is because r_g is usually much smaller than a risk-free zero-coupon bond rate; in other words, $e^{-r_g(T-t)} >> P(t, T)$. Although $P(t, T)$ is stochastic, in general it will never rise to the level of $e^{-r_g(T-t)}$, because of the small value usually taken by r_g.

Whether one considers a constant or stochastic interest rate guarantee, the default time, in our continuous setting, is always defined as the first time the assets A cross L^g (the default barrier described by one of the above expressions [3.1] or [3.2]), so

$$\tau = \inf\{s \in [0,T], A_s \le L_s^g\}.$$

One immediately obtains the generic no-arbitrage price of a participating contract under the risk-neutral probability:

$$\tilde{V}_0 = \mathbb{E}_Q[e^{-\int_0^T r_s\, ds}\ (L_T^g + \delta(\alpha A_T - L_T^g)^+)\ \mathbf{1}_{\tau > T} + e^{-\int_0^\tau r_s\, ds}\ A_\tau \mathbf{1}_{\tau \le T}], \tag{3.3}$$

where $A_\tau = L_\tau^g$ when the assets are supposed continuous (as is the case in this paper). Clearly, if $\tau > T$, default did not happen, and the payoff $L_T^g + \delta(\alpha A_T - L_T^g)^+$, corresponding to the minimum guarantee plus the dividend (or "participation"), is paid at the maturity of the contract. The situation $\tau \le T$ describes either $\tau = T$, default at maturity, or $\tau < T$, early default. Restricting oneself to default at maturity reduces to a Merton model, and then correspondingly formula (3.3) simplifies to (2.3). What we want to study is the impact and modeling of the condition $\tau \le T$. In this state we suppose that the rebate L_τ^g is paid upon bankruptcy, at the random stopping time τ. This justifies the introduction of the second term in formula (3.3).

When the guaranteed rate is constant, as with (3.1), and under a Vasicek specification of r, one can price (3.3) semi-explicitly as shown by Bernard et al. (2005, see section 2.1, pp. 507–508). With a stochastic guaranteed rate, as in (3.2), formula (3.3) can be priced in closed form (see calculation of [2.4] in Bernard et al. 2006, p. 182). Recall that one issue with the Vasicek interest rate model is that it admits negative interest rates (although it happens with a very small probability).

The setting detailed here models and prices participating policies as it would do with exotic bonds. Yet we are faced with the question of actuarial practices and safety loadings. The following paragraphs describe how to protect life insurance companies and policyholders, in a Black-Cox-Vasicek framework.

3.2 Continuously Protected Policies (Early Default Setting)

In the early default setting, pricing the default put is a complex path-dependent problem. Indeed, two difficulties arise. The first one is technical and related to the intrinsic valuation of path-dependent exotic options. The second one is financial and, in fact, multiple: is the company audited continuously or discretely (at the end of each year for instance)? Has the company to protect itself discretely or continuously between 0 and T? How does it choose to protect itself and in what proportion? We start our analysis by considering the case where default can happen continuously (at any time between 0 and T), and where the company aims at buying a continuous protection.

The value of a fully protected (continuously between 0 and T, and therefore also at T) participating contract is always worth

$$\hat{V}_0 = \mathbb{E}_Q[e^{-\int_0^T r_s\,ds}\,(L_T^g + \delta(\alpha A_T - L_T^g)^+)]. \tag{3.4}$$

Theoretically, the price of the total continuous protection (denoted hereafter by G) can be evaluated very easily. Indeed, it suffices to compute the difference between the prices of the protected and the unprotected policies. The total continuous protection price is therefore the difference of (3.4) and (3.3), which can be readily expressed as

$$G_0 = \hat{V}_0 - \tilde{V}_0 = \mathbb{E}_Q[(e^{-\int_0^T r_s\,ds}\,(L_T^g + \delta(\alpha A_T - L_T^g)^+) - e^{-\int_0^\tau r_s\,ds}\,L_\tau^g)\,\mathbf{1}_{\tau\le T}]. \tag{3.5}$$

When the barrier is stochastic and defined as in (3.2), formula (3.5) can be evaluated in closed form (see calculation of [2.4] in Bernard et al. 2006, p. 182). Working under this assumption, we display our results in Table 5.

Because the framework is unchanged, the protected policy's price, $\hat{V}_0$, is still worth 92.42 (see Table 3 for a comparison with previous results). It is interesting to note that $\tilde{V}_0 = 91.34$, the value of the contract that is risky between 0 and T, is bigger than the value $V_0 = 90$ of the contract that is risky only at time T. Why would an apparently riskier contract (because of a possible default between 0 and T) be worth more than an apparently less risky contract (defaultable only at maturity T)? The answer is simple: early default limits the losses incurred by the company and the insured. This is why the premium $G_0 = 1.08$ is (less than half) smaller than the premium $S_0 = 2.42$. Indeed, in the Black and Cox setting, because the company is immediately in bankruptcy, the insured recover the guaranteed amount at the time of default τ and suffer more from a wasted opportunity (of continuing up to T and potentially receiving a dividend) than from a real loss. It should also be noticed that because r_g is

Table 5
Results

$\tilde{V}_0$	$\hat{V}_0$	G_0
91.34	92.42	1.08

inferior to r, it is not desirable for investors to recover the guaranteed rate compared to an investment in risk-free bonds.

3.3 Discretely Protected Policies (Early Default Case)

Assume that the balance sheet of the company is monitored at the end of every year: default can be declared only discretely on this set of dates. Therefore, the main concern of the managers of the company is to avoid shortfalls of the assets at the end of each year.

An initial idea is to buy as many puts as there are years in the contract's tenor. This is the simplest way for the company to ensure that it will be solvent at every end of year: each time, its assets A must be over the minimum guaranteed amount (i.e., $A_{t_i} > L^g_{t_i}$). The discounted payoffs of the protection just defined (being a simple series of put options) can be represented as

$$e^{-\int_0^{t_1} r_s\,ds}\,(L^g_{t_1} - A_{t_1})^+ + e^{-\int_0^{t_2} r_s\,ds}\,(L^g_{t_2} - A_{t_2})^+ + \cdots + e^{-\int_0^{t_n} r_s\,ds}\,(L^g_{t_n} - A_{t_n})^+.$$

Consider for the sake of example the ith put. It admits the following characteristics: a maturity t_i, a strike $L^g_{t_i}$, a final payoff $(L^g_{t_i} - A_{t_i})^+$, and its underlying assets are, of course, A. This put can be evaluated in closed form very easily. For instance, the value of the representative ith put with a maturity t_i:

$$\mathbb{E}_Q[e^{-\int_0^{t_i} r_s\,ds}\,(L^g_{t_i} - A_{t_i})^+] = P(0,T)L^g_T\,\mathrm{N}(d(\mathrm{t_i})) - \mathrm{A_0}\mathrm{N}(d(\mathrm{t_i}) - \sqrt{\xi(\mathrm{t_i})}), \tag{3.6}$$

where

$$d(t_i) = \frac{\ln\left(\dfrac{P(0,T)L^g_T}{A_0}\right) + \dfrac{1}{2}\,\xi(t_i)}{\sqrt{\xi(t_i)}} \text{ and } \xi(u) = \int_0^u[(\sigma_P(s,T) + \rho\sigma)^2 + \sigma^2(1 - \rho^2)]\,ds.$$

Parameters are chosen as in Table 2. The company buys as many annual puts as there are years left in the contract life: that is, the company protects itself from default at each year end. In this situation the protection is very expensive and is equal to 6.85. Indeed, it is redundant: all the puts cover the first period (0 to t_1), all the puts except the first one cover the second period (t_1 to t_2), and so on. A more refined strategy is necessary to protect the life insurance company in the context of discrete monitoring. In essence, the appropriate protection has to be path dependent. Indeed, the discounted payoff of such a protection can be defined as

$$e^{-\int_0^{t_1} r_s\,ds}\,(L^g_{t_1} - A_{t1})^+ + e^{-\int_0^{t_2} r_s\,ds}\mathbf{1}_{A_{t_1}>L^g_{t_1}}\,(L^g_{t_2} - A_{t2})^+ + \cdots + e^{-\int_0^{t_n} r_s\,ds}\,\mathbf{1}_{A_{t_1}>L^g_{t_1}\ldots A_{t_{n-1}}>L^g_{t_{n-1}}}\,(L^g_{t_n} - A_{t_n})^+,$$

and the associated price can be computed by means of Monte Carlo simulations.

In the context of discrete monitoring, a surprise can happen at the end of a particular year, meaning that $A_{t_i} << L^g_{t_i}$. On average, the surprise will be less flagrant than waiting for the maturity T (the Merton case). Recall also that in a continuous monitoring situation, no surprise can happen (Black and Cox case, and considering diffusive assets, of course). The first conclusion is that the price of the protection under discrete monitoring should be intermediate between the ones under continuous monitoring and terminal (at maturity) monitoring. The second conclusion is that because surprises are possible under discrete monitoring, it should be possible to introduce a set of parameters ψ_i, in full analogy with the developments of the default at maturity case. Thus, the ideas of the mixed framework can be extended transparently and directly from the Merton case to the discrete monitoring one. For the sake of brevity, we do not repeat the same procedure.

4. Protection in Practice

A question that often arises is: How can the protection be constructed using market instruments? More precisely, can we find options, swaps, or other similar products, in order to directly build the default put and protect the company and the insured? This is the question that we address at present. We also study the impact of using market instruments on the ruin probability, and on the severity of the ruin that the company can incur.

4.1 Construction of the Protection

Very often the vanilla or slightly exotic options that a life insurance company can consider buying are short maturity products, typically with a one-year maturity. This is clearly not the horizon of an insurance company (we do not consider the possibility of rolling over one-year-maturity derivative positions). On the other hand, swaps and swaptions are long-term products, but they do not necessarily possess payoffs directly meeting the needs of insurance companies.

However, a class of products emerged a couple of years ago that possess excellent characteristics with respect to the problem at hand. These products are called equity default swaps. They were created by JP Morgan in 2000 and are, in fact, insurance policies on equity. Indeed, equity default swaps (EDSs) were created for similar reasons as credit default swaps (CDSs). They provide protection against a severe equity decline, whereas CDSs provide protection against credit events on a corporate bond. Note that an equity fall of x% is a well-identified event, whereas credit events are sometimes subject to controversy. EDSs share with CDSs the denomination ‘‘swap’’ because the investor pays his or her fee in installments rather than as a lump sum. Typical payment periods are six months for EDSs and three months for CDSs. Typical maturities are five years for both products. The other leg of the swap is the payment to the investor of a rebate when the critical event happens: when the stock loses x% of its initial value—where x is fixed contractually—for the EDS, or when a credit event occurs for the CDS. EDSs are structured so as to ensure very severe drawdowns of the underlying stock: a barrier at 70% of the initial stock value is commonplace. If this event happens, a rebate, usually 50% of the loss from the initial stock value, is paid to the investor, and, of course, installments are ceased. As far as CDSs are concerned, the rebate compensating for the underlying bond's depreciation upon default is proportional to the loss incured.

Note that another way to seek protection is to consider a dynamic investment strategy. Much work has been devoted to guaranteed funds; see, for example, Gerber and Pafumi (2000), Basak (1995), and El Karoui et al. (2005). A typical and popular strategy that can be used is the so-called CPPI, which stands for constant proportion portfolio insurance (see Black and Perold 1992).

4.2 Impact of the Protection

We study next the impact of the chosen protection (investment of the safety loading) on the probability and severity of ruin of the insurance company. We consider for this illustration three different settings, denoted by ‘‘a,’’ ‘‘b,’’ and ‘‘c.’’ Setting a is a theoretical setting: it corresponds to the mixed approach studied in this article where the payoff $\psi L_T^g + (1 - \psi)A_T$ is recovered by policyholders in case of default. In setting a, it is assumed that the levied safety loading is invested in an ad hoc put option that can be found in the market. Setting b explores a natural idea that can be found, for example, in Ballotta, Esposito, and Haberman (2006). This idea is as follows: the safety loading is simply invested in the fund assets at time 0. Finally, setting c describes a protection by means of equity default swaps. To simplify the exposition, computations are done under deterministic interest rates, the maturity of contracts is five years in all three settings, and the EDS premium is paid at inception and not by installments.

Note that settings a and b are referred to the case in which the solvency condition is checked only at maturity, while setting c protects policyholders in case of early default. Note also that all the computations performed here are done in the historical world. Reasonably, one is interested in real-world ruin probabilities and real-world losses. Finally, in the following developments, we consider situations where the fair price at time zero of a specified type of protection is charged to the insured, and we study the impact of a change of volatility on these situations.

Let us now give the ruin probabilities and expected losses formulas under the three subsettings:

Setting a: The ruin probability is simply $P(A_T < L_T^g)$.

In case of ruin, the loss incurred by the insured, which can be called severity of ruin, is $L_T^g - (\psi L_T^g + (1 - \psi)A_T) = (1 - \psi)(L_T^g - A_T)$. This is the distance from the barrier, at default. For comparison purposes, we define a severity of ruin that is expected and discounted at time 0. So our severity of ruin indicator will be defined by

$$E_P(e^{-rT}(1 - \psi)(L_T^g - A_T)\mathbf{1}_{A_T<L_T^g}).$$

Setting b: In this setting, the safety loading $\hat{\hat{S}}_0$ is invested in the assets at time 0. So the asset process starts at $A'_0 = A_0 + \hat{\hat{S}}_0$ (this is the approach of Ballotta et al. 2006). A' constitutes the total assets owned by the insurance company.

The probability of ruin becomes $P(A'_T < L_T^g)$. The loss upon default is worth $L_T^g - A'_T$, yielding a severity of ruin estimate from time 0:

$$E_P(e^{-rT}(L_T^g - A'_T)\mathbf{1}_{A'_T<L_T^g}).$$

Setting c: In this setting, the safety loading $\hat{\hat{S}}_0$ is invested in equity default swaps. This is an intertemporal setting, when a and b are not. The underlying, U, of an EDS is supposed to be representative of the assets of the insurance company, and proportional to them, namely, $\forall t\ U_t = \zeta A_t$ where for a typical insurance company $0 < \zeta << 1$. We suppose that the insurance company buys ϕ EDSs (the case $\phi = \dfrac{1}{\zeta}$ is naturally the complete hedge of A_0, so a value superior to what may be required).

Let us now give the payoff of the EDS position. As shown above, an EDS typically pays $(U_0 - 0.7U_0)/2 = 0.15U_0$ and terminates at the first time τ such that $U_\tau = 70\%\ U_0$. If the underlying does not touch the barrier set at 70% of its initial value, the contract terminates with null value at maturity (the maturity of the EDS is set equal to T, maturity of the contracts issued by the company). The no-arbitrage price of the EDS position is therefore

$$\phi\ E_Q(0.15U_0e^{-r\tau}\mathbf{1}_{\tau<T}),$$

where $\tau = \inf\{t < T | U_t \leq 0.7U_0\}$.

A simple proportionality argument yields $\tau = \inf\{t < T | A_t \leq 0.7A_0\}$. As concerns ϕ, it naturally satisfies

$$\hat{\hat{S}}_0 = \phi\ E_Q(\zeta 0.15A_0e^{-r\tau}\mathbf{1}_{\tau<T}). \tag{4.1}$$

In the present situation, EDSs are bought to limit the severity of ruin beyond a certain level. Ruin can occur in two different manners: at time τ if the company's assets suffer from a severe drawdown (in this situation EDSs a reactivated), or at time T if the assets never touch the barrier but nevertheless end below L_T^g (in this situation EDSs are not activated). This yields the following ruin probability:

$$P(\tau < T) + P(\tau \geq T, A_T < L_T^g).$$

The severity of ruin indicator can be constructed as

$$E_P(e^{-r\tau} \max([L_\tau^g - (0.7A_0 + 0.15\phi\zeta A_0)],0)\ \mathbf{1}_{\tau<T}) + E_P(e^{-rT}\ [L_T^g - A_T]^+\ \mathbf{1}_{\tau\geq T}), \tag{4.2}$$

where $\phi\zeta$ is calibrated from the safety loading (see eq. [4.1]) as follows:

$$\phi\zeta = \frac{\hat{\hat{S}}_0}{E_Q(0.15A_0e^{-r\tau}\mathbf{1}_{\tau<T})}.$$

Let us illustrate these three settings with a numerical example. For the sake of simplicity, it is assumed that the assets are made of stocks (in the case of a mix of stocks and bonds, the protection in setting c would use both EDSs and CDSs). We consider a contract whose fair price is $V_0 = 90$; a safety loading equal to 1 is charged. The initial total premium is thus 91.

Table 6 gives the parameters used in our illustration. Here ψ is computed based on (2.13). The value of the put option being 1.21 (based on the computation of the related risk-neutral expression), and

Table 6
Model Parameters

A_0	T	α	μ	r	$\hat{\hat{S}}_0$	r_g
100	5	0.9	6.5%	5%	1	3.5%

the safety loading being set to 1, one readily has $\psi = 0.82$. As far as δ is concerned, it is set in order to make the contract fair and therefore depends on the level of the volatility. Because the computations are done in the historical world, it is necessary to specify the drift of the assets in the real world; here we chose $\mu = 6.5\%$.

In Figure 1 we represent the ruin probability with respect to the volatility in the three settings, and in Figure 2 we take a look at the severity of ruin, also with respect to the company's asset volatility. The amount of safety loading is, of course, the same in the three settings.

We observe from the graphs that ruin probabilities are comparable in the two settings a and b, but that ruin severities are always lower when using default puts instead of investing the safety loading in additional securities.

Let us now take a look at setting c. The ruin probability is higher using EDSs than with other methods. However, one observes the following interesting feature: the ruin severity could be smaller with EDSs than with a reinvestment of the safety loading in the assets. Figures 1 and 2 seem to suggest that the pattern of setting a, b, and c are relatively close for low volatility, but when the volatility is high, the severity of ruin in setting c is highest. To conclude, the smaller levels of ruin probability and severity are mostly obtained with a protection made of put options. In case these put options are not available or cannot be synthesized in the market, two situations arise. If the ruin probability is the indicator to be minimized, then one should reinvest the safety loading in the assets, as in Ballotta et al. (2006). If one is interested in minimizing the severity of ruin, then investing in EDSs could be profitable. Note, however, that whatever the setting (and if charging only partly the default put), it will not be possible to avoid ruin with certainty.

Figure 1
***P*($\tau \leq$ T) with Respect to σ**

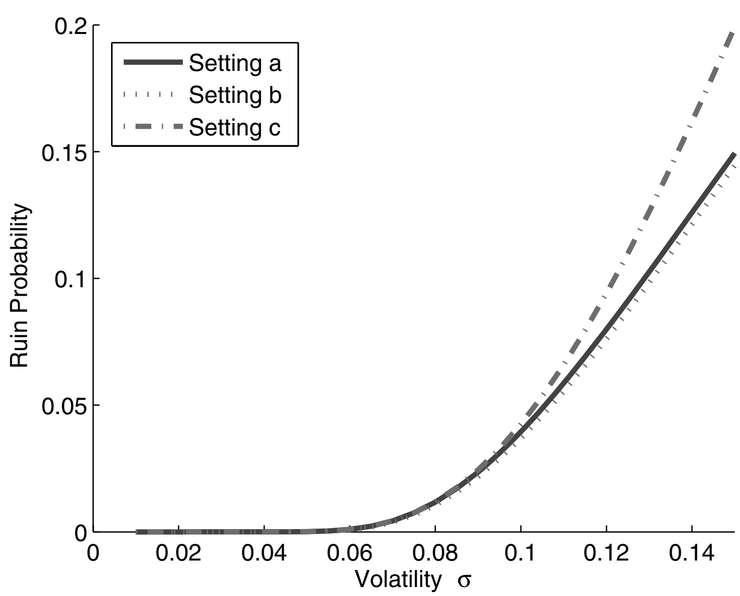
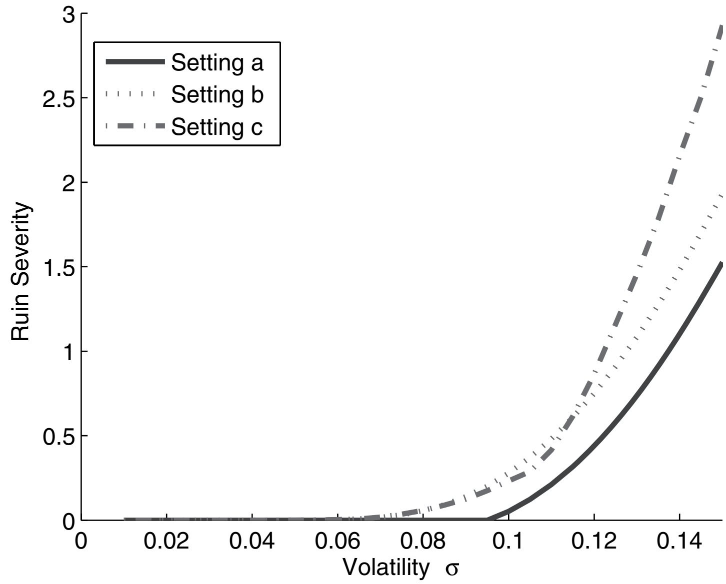


Figure 2
Ruin Severity with Respect to σ



5. Summary and Conclusions

This study is devoted to the calculation of appropriate premia and loadings for participating insurance contracts. We introduce safety loadings in close relationship to default puts on insurance companies. These loadings reflect the asset and credit risks of underlying products. This study also explains why different insurers sell similar contracts at different prices (the difference being a credit risk premium). Loadings may depend on various features, such as the preference of the insurer or the insured, regulation, enterprise risk management, ratings, and credit risk.

We believe that the approach developed in this article can be applied in other fields, like bank deposit insurance. Indeed, Merton (1977, 1978) showed that bank deposit guarantees are equivalent to default puts on the assets of the bank hosting the deposits. In his first article, there is one final date for monitoring, while in the second monitoring can occur at any time and is driven by a Poisson distribution. There are clear analogies between the guarantees of bank deposits and the guarantees attached to contracts like the ones studied in this article. Our conclusion—via this example of bank deposit guarantees—is that the developments within this article can be of interest to other subfields of finance and insurance.

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Appendix

Computation of the Expressions of A_T and $P(\tau, T)$

We use a one-factor Heath, Jarrow, and Morton interest rate model with a deterministic volatility for the T-zero-coupon bond of an exponential type (this is the Hull and White choice). With $\nu > 0$ and $a > 0$, the volatility structure is

$$\sigma_P(t,T) = \frac{\nu}{a}(1 - e^{-a(T-t)}). \tag{A.1}$$

Under the Risk-Neutral Probability *Q*

The dynamics of the zero-coupon bond $P(u, t)$ price with expiry time t under the risk-neutral probability Q can be written like

$$\frac{dP(u, t)}{P(u, t)} = r_u\, du - \sigma_P(u, t)\, dZ_1^{Q}(u),$$

where $Z_1^{Q}(t)$ is a Q-Brownian motion.

We apply Itô's Lemma to the function $f(X, Y) = \ln\left(\frac{X}{Y}\right)$, where $X = P(u, T)$ and $Y = P(u, t)$ because we know both of the dynamics of X and Y under Q. One shows that

$$df = -\frac{1}{2}(\sigma_P^2(u, T) - \sigma_P^2(u, t))\, du - (\sigma_P(u, T) - \sigma_P(u, t))\, dZ_1^{Q}(u).$$

Then integrating from 0 to t, we obtain (thanks to the relation $P(t, t) = 1$)

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp\left(-\frac{1}{2}\int_0^t (\sigma_P^2(u, T) - \sigma_P^2(u, t))\, du - \int_0^t (\sigma_P(u, T) - \sigma_P(u, t))\, dZ_1^{Q}(u)\right). \tag{A.2}$$

Under the Forward-Neutral Probability Q_T

Let us now denote by Q_T the T-forward-neutral measure. It is defined through its Radon-Nikodym derivative:

$$\frac{dQ_T}{dQ} = e^{-\int_0^T \sigma_P(s,T)\, dZ_1^{Q}(s) - \frac{1}{2}\int_0^T \sigma_P^2(s,T)\, ds}.$$

From the Girsanov theorem the process $Z_1^{Q_T}$ defined by $dZ_1^{Q_T} = dZ_1^{Q} + \sigma_P(t, T)\, dt$ is a Q_T-Brownian motion. The process $Z_2^{Q_T}$ is then built such that $Z_1^{Q_T}$ and $Z_2^{Q_T}$ are Q_T-noncorrelated standard Brownian motions.

Under the forward-neutral probability measure Q_T, the assets value, A_t, and the zero-coupon bond, $P(t, T)$, follow the stochastic diffusions

$$\frac{dP(t, T)}{P(t, T)} = (r_t + \sigma_P^2(t, T))\, dt - \sigma_P(t, T)\, dZ_1^{Q_T} \tag{A.3}$$

and

$$\frac{dA_t}{A_t} = (r_t - \sigma\rho\sigma_P(t, T))\, dt + \sigma(\rho dZ_1^{Q_T} + \sqrt{1 - \rho^2}\, dZ_2^{Q_T}). \tag{A.4}$$

We replace $dZ_1^{Q}(u)$ by $dZ_1^{Q_T}(u) = dZ_1^{Q}(u) + \sigma_P(u, T)\, du$ in equation (A.2) to obtain the following expression for $P(t, T)$:

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp\left(\frac{1}{2}\int_0^t (\sigma_P(u, T) - \sigma_P(u, t))^2\, du - \int_0^t (\sigma_P(u, T) - \sigma_P(u, t))\, \mathrm{d}Z_1^{Q_T}(u)\right). \tag{A.5}$$

We integrate (A.3) and (A.4) under Q_T, applying Itô's Lemma to $f(X, Y) = \ln\left(\frac{X}{Y}\right)$, where $X = A_u$ and $Y = P(u, T)$, we obtain

$$\frac{A_t}{P(t,T)} = \frac{A_0}{P(0,T)} \exp\left(\int_0^t (\sigma_P(u,T) + \rho\sigma)\, dZ_1^{Q_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\, dZ_2^{Q_T}(u) - \frac{1}{2}\int_0^t ((\sigma_P(u,T) + \rho\sigma)^2 + \sigma^2 - \rho^2\sigma^2)\, du\right). \quad \text{(A.6)}$$

Finally, we replace the expression of $P(t, T)$ given by equation (A.5) in the formula (A.6), we have the result we want to prove:

$$A_t = \frac{A_0}{P(0,t)} \exp\left(\int_0^t (\sigma_P(u,t) + \rho\sigma)\, dZ_1^{Q_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\, dZ_2^{Q_T}(u) + \int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t) + \rho\sigma) + \frac{\sigma_P^2(u,t) - \sigma^2}{2}\right) du\right).$$

COMPUTATION OF THE DISCRETE PROTECTION

Here we prove formula (3.6):

$$\begin{aligned}
\mathbb{E}_Q[e^{-\int_0^{t_i} r_s\, ds}\, (L_{t_i}^g - A_{t_i})^+] &= \mathbb{E}_Q\left[e^{-\int_0^{t_i} r_s\, ds}\, P(t_i, T)\left(L_T^g - \frac{A_{t_i}}{P(t_i,T)}\right)^+\right] \\
&= P(0,T)\, \mathbb{E}_{Q_T}\left[\frac{P(t_i,T)\left(L_T^g - \frac{A_{t_i}}{P(t_i,T)}\right)^+}{P(t_i,T)}\right] \\
&= P(0,T)\, \mathbb{E}_{Q_T}\left[\left(L_T^g - \frac{A_{t_i}}{P(t_i,T)}\right)^+\right].
\end{aligned}$$

Recalling that L_T^g is a constant and that $\frac{A_{t_i}}{P(t_i,T)}$ can be cast in the form $\frac{A_0}{P(0,T)}\, e^{R_u - (<R>u/2)}$, where R is a martingale under Q_T and $< R >$ its quadratic variation, we see that this put can be evaluated in closed form very easily as shown below.

Let us define by ε:

$$\varepsilon = P(0,T)\mathbb{E}_{Q_T}\left[\left(L_T^g - \frac{A_{t_i}}{P(t_i,T)}\right)^+\right].$$

Note that

$$\frac{A_u}{P(u,T)} = \frac{A_0}{P(0,T)}\, e^{R_u - (1/2)\xi(u)}, \quad \text{(A.7)}$$

where the differential of R is defined by

$$dR_s = (\sigma_P(s,T) + \rho\sigma)\, dZ_1^{Q_T}(s) + \sigma\sqrt{1-\rho^2}\, dZ_2^{Q_T}(s), \quad \text{(A.8)}$$

and the quadratic variation of R is

$$\xi(u) = < R >_u = \int_0^u [(\sigma_P(s,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)]\, ds. \quad \text{(A.9)}$$

We prove below how to obtain the following closed form of ε using Girsanov's theorem and time change techniques:

$$\varepsilon = P(0, T)L_T^g N\left(\frac{\ln\left(\frac{P(0, T)L_T^g}{A_0}\right) + \frac{1}{2}\xi(t_i)}{\sqrt{\xi(t_i)}}\right) - A_0 N\left(\frac{\ln\left(\frac{P(0, T)L_T^g}{A_0}\right) - \frac{1}{2}\xi(t_i)}{\sqrt{\xi(t_i)}}\right). \quad (A.10)$$

Let us first write ε as

$$\varepsilon = P(0, T)\left(L_T^g Q_T\left(\frac{A_{t_i}}{P(t_i, T)} < L_T^g\right) - \mathbb{E}_{Q_T}\left[\frac{A_{t_i}}{P(t_i, T)}\mathbf{1}_{\{(A_{t_i}/P(t_i,T))<L_T^g\}}\right]\right).$$

The key to the computation is the Dubins-Schwarz theorem (time change technique), which states that there exists a unique Q_T-Brownian motion B such that

$$\forall u \in [0,T],\ R_u = R_0 + B_{\xi(u)}. \quad (A.11)$$

Using this representation theorem, we get a new expression of the two parts of the expression ε:

$$Q_T\left(\frac{A_{t_i}}{P(t_i, T)} < L_T^g\right) = Q_T\left(R_{t_i} - \frac{1}{2}\xi(t_i) < \ln\left(\frac{P(0,T)L_T^g}{A_0}\right)\right),$$

$$\mathbb{E}_{Q_T}\left[\frac{A_{t_i}}{P(t_i, T)}\mathbf{1}_{\{(A_{t_i}/P(t_i,T))<L_T^g\}}\right] = \frac{A_0}{P(0, T)}\mathbb{E}_{Q_T}\left[e^{R_{t_i}-(1/2)\xi(t_i)}\mathbf{1}_{\{R_{t_i}-(1/2)\xi(t_i)<\ln\ (P(0,T)L_T^g/A_0)\}}\right].$$

Because $R_{t_i} - \frac{1}{2}\xi(t_i) = B_{\xi(t_i)} - \frac{1}{2}\xi(t_i)$ is a normal variable, it is then straightforward to obtain the formula (A.10) for ε.

REFERENCES

ASMUSSEN, S., N. MUNKEGADE, D. MADAN, AND M. PISTORIUS. 2007. Pricing Equity Default Swaps under an Approximation to the CGMY Lévy Model. *Journal of Computational Finance* 11(2): 79–93.

BALLOTTA, L. 2005. A Lévy Process-Based Framework for the Fair Valuation of Participating Life Insurance Contracts. *Special Issue of Insurance: Mathematics and Economics* 37(2): 173–196.

BALLOTTA, L., G. ESPOSITO, AND S. HABERMAN. 2006. The IASB Insurance Project for Life Insurance Contracts: Impact on Reserving Methods and Solvency Requirements. *Insurance: Mathematics and Economics* 39(3): 356–375.

BALLOTTA, L., S. HABERMAN, AND N. WANG. 2006. Guarantees in With-Profit and Unitised With-Profit Life Insurance Contracts: Fair Valuation Problem in Presence of the Default Option. *Journal of Risk and Insurance* 73(1): 97–121.

BARTH, M. 2000. A Comparison of Risk-Based Capital Standards under the Expected Policyholder Deficit and the Probability of Ruin Approach. *Journal of Risk and Insurance* 67(3): 397–414.

BASAK, S. 1995. A General Equilibrium Model of Portfolio Insurance. *Review of Financial Studies* 8: 1059–1090.

BERNARD, C., O. LE COURTOIS, AND F. QUITTARD-PINON. 2005. Market Value of Life Insurance Contracts under Stochastic Interest Rates and Default Risk. *Insurance: Mathematics and Economics* 36(3): 499–516.

BERNARD, C., O. LE COURTOIS, AND F. QUITTARD-PINON. 2006. Development and Pricing of a New Participating Contract. *North American Actuarial Journal* 10(4): 179–195.

BERNARD, C., AND C. LEMIEUX. 2008. Fast Simulation on Equity-Linked Life Insurance Contracts with a Surrender Option. *Proceedings of the 2008 Winter Simulation Conference*.

BLACK, F., AND J. C. COX. 1976. Valuing Corporate Securities: Some Effects of Bond Indenture Provisions. *Journal of Finance* 31(2): 351–367.

BLACK, F., AND A. PEROLD. 1992. Theory of Constant Proportion Portfolio Insurance. *Journal of Economic Dynamics and Control* 16: 403–426.

BOYLE, P. P., AND E. S. SCHWARTZ. 1977. Equilibrium Prices of Guarantees under Equity-Linked Contracts. *Journal of Risk and Insurance* 44: 639–660.

BOYLE, P. P., AND W. TIAN. 2008. The Design of Equity-Linked Contracts. *Insurance: Mathematics and Economics* 43(3): 305–315.

BRENNAN, M. J., AND E. S. SCHWARTZ. 1976. The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee. *Journal of Financial Economics* 3: 195–213.

BRIYS, E., AND F. DE VARENNE. 1994. Life Insurance in a Contingent Claim Framework: Pricing and Regulatory Implications. *Geneva Papers on Risk and Insurance Theory* 19(1): 53–72.

BRIYS, E., AND F. DE VARENNE. 1997. On the Risk of Insurance Liabilities: Debunking Some Common Pitfalls. *Journal of Risk and Insurance* 64(4): 673–694.

BRIYS, E., AND F. DE VARENNE. 2001. *Insurance from Underwriting to Derivatives*. New York: Wiley Finance.

BÜHLMANN, H. 2002. New Maths for Life Actuaries. *ASTIN Bulletin* 32(2): 209–211.

BÜHLMANN, H. 2004. Multidimensional Valuation. *Revue Finance, Insurance and Finance Special Issue* 25: 15–29.

COWLEY, A., AND J. D. CUMMINS. 2005. Securitization of Life Insurance Assets and Liabilities. *Journal of Risk and Insurance* 72(2): 193–226.

CUMMINS, J. D. 2000. Allocation of Capital in the Insurance Industry. *Risk Management and Insurance Review* 3: 7–27.

CUMMINS, J. D., AND D. W. SOMMER. 1996. Capital and Risk in Property-Liability Insurance Markets. *Journal of Banking and Finance* 20: 1069–1092.

EL KAROUI, N., M. JEANBLANC, AND V. LACOSTE. 2005. Optimal Portfolio Management with American Capital Guarantee. *Journal of Economic Dynamics and Control* 29: 449–468.

FROOT, K. 2007. Risk Management, Capital Budgeting, and Capital Structure Policy for Insurers and Reinsurers. *Journal of Risk and Insurance* 74(2): 273–300.

GERBER, H., AND G. PAFUMI. 2000. Pricing Dynamic Investment Fund Protection. *North American Actuarial Journal* 4(2): 28–41.

HARDY, M. R. 2003. *Investment Guarantees: Modelling and Risk Management for Equity-Linked Life Insurance*. New York: Wiley.

LONGSTAFF, F. A., AND E. S. SCHWARTZ. 1995. A Simple Approach to Valuing Risky Fixed and Floating Rate Debt. *Journal of Finance* 50(3): 789–820.

MERTON, R. C. 1974. On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance* 29: 449–470.

MERTON, R. C. 1977. An Analytic Derivation of the Cost of Deposit Insurance and Loan Guarantees. *Journal of Banking and Finance* 1: 3–11.

MERTON, R. C. 1978. On the Cost of Deposit Insurance When There Are Surveillance Costs. *Journal of Business* 51: 439–452.

SCHWEIZER, M. 2001. From Actuarial to Financial Valuation Principles. *Insurance: Mathematics and Economics* 28: 31–47.

SOMMER, D. W. 1996. The Impact of Firm Risk on Property-Liability Insurance Prices. *Journal of Risk and Insurance* 63(3): 501–514.

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