Financial Bounds for Insurance Claims

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Background & Objectives

- (*"Explicit Representation of Cost-efficient Strategies"* with Phelim Boyle (Wilfrid Laurier University))
- Main Result of this paper: Provide the **cheapest** and the **most expensive** strategy using the financial market to achieve a given probability distribution
  $\Rightarrow$ **bounds on prices of financial claims with a given cdf.**
- Our main objective:
  1. To find bounds on prices of claims
     - that cannot be hedged perfectly in the market.
     - but for which we know the cdf under the physical probability.
  2. when the pricing is "market-consistent"

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Some Assumptions on the Financial Market

- Consider an arbitrage-free and complete market. Any financial claim has a unique price $c(X_T)$ (price of the replicating strategy).
- Given a strategy with payoff X_T at time T, there exists Q, such that its price at 0 is

$$c(X_T) = \mathbb{E}_Q[e^{-rT}X_T]$$

- P ("physical measure") and Q ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T, \quad \mathbf{c(X_T)} = \mathbb{E}_Q[e^{-rT}X_T] = \mathbb{E}_{\mathbf{P}}[\xi_{\mathbf{T}}\mathbf{X_T}].$$

Assumptions on Preferences

Denote by X_T the final wealth of the investor and $U(X_T)$ the objective function of the agent.

1. Market participants all have a fixed investment horizon $T > 0$ and there is no intermediate consumption (one-period model).
2. **Agents' preferences depend only on the probability distribution of terminal wealth:** "law-invariant" preferences. (if $X_T \sim Z_T$ then: $U(X_T) = U(Z_T)$.)
3. **Agents prefer "more to less"**: if c is a non-negative random variable $U(X_T + c) \geqslant U(X_T)$.
4. **Agents are risk-averse**:

$$\begin{cases} E[X_T] = E[Y_T] \\ \forall d \in \mathbb{R}, E[(X_T - d)^+] \leq E[(Y_T - d)^+] \end{cases} \Rightarrow U(X_T) \geqslant U(Y_T)$$

Bid and Ask prices for insurance claims in the *absence* of a financial market using "certainty equivalents"

Investing in a bank account is the only investment.

- From the **viewpoint of the insured** with objective function $U(\cdot)$ and initial wealth ω the (bid) price, p^b,

$$U[(\omega - p^b)e^{rT}] = U[\omega e^{rT} - C_T].$$

- From the **viewpoint of the insurer** with a given objective function $V(\cdot)$ and initial wealth ω the ask price, p^a,

$$V[(\omega + p^a)e^{rT} - C_T] = V[\omega e^{rT}].$$

Properties

1. Bid and Ask prices verify

$$p_\bullet \geqslant e^{-rT}\mathbb{E}_P[C_T].$$

(no undercut principle)

2. If the insurer is risk neutral ($v(x) = x$), then

$$p_b \geqslant p_a = e^{-rT}\mathbb{E}_P[C_T]$$

3. In the case of exponential utility $p_a = p_b$.
4. In the case of Yaari's theory $p_a = p_b$.
5. In general, nothing can be said. $u(x) = v(x) = 1 - 1/x$, both agents have same initial wealth, $C_T \sim U(0,2)$. See next figure.

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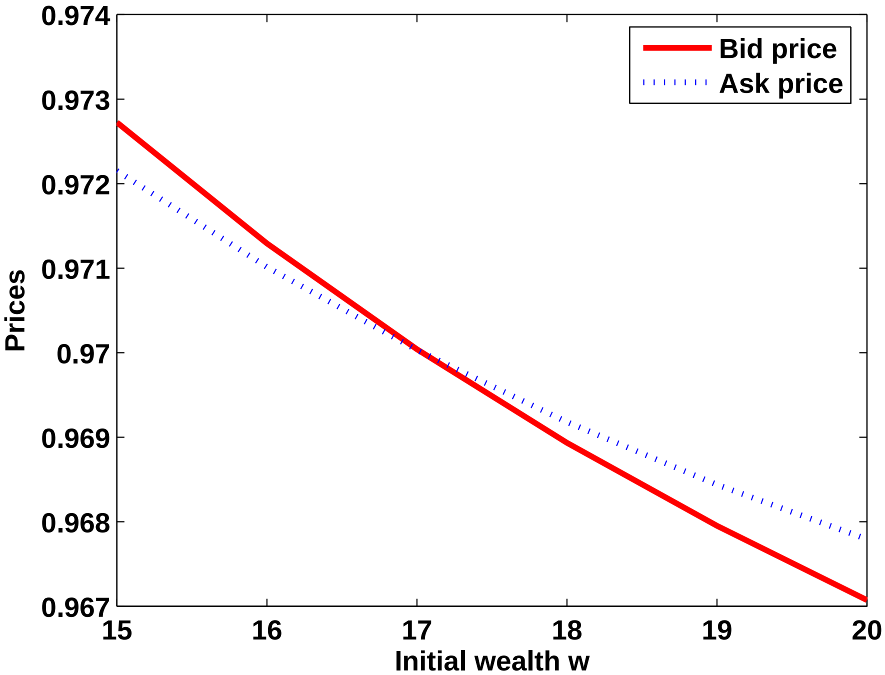

Bid price
Ask price
0.974
0.973
0.972
0.971
0.97
0.969
0.968
0.967
Prices
15
16
17
18
19
20
Initial wealth w

In the *presence* of a financial market

In the *presence* of a financial market, it is now possible to trade in a risky asset.

Let $A(\omega)$ be the set of random wealth X_T that

- can be generated with an initial budget of $\omega > 0$
- using an "admissible" trading strategy (self financing and adapted)

In the *absence* of a financial market, there is only one possible final wealth

$$X_T = \omega e^{rT}$$

so that $A(\omega) = \{\omega e^{rT}\}$.

Bid and Ask prices in the *presence* of a financial market

- From the **viewpoint of the insured** with objective $U(\cdot)$ and initial wealth ω the (bid) price, p^b, follows from

$$\sup_{X_T \in A(\omega - p^b)} \{U[X_T]\} = \sup_{X_T \in A(\omega)} \{U[X_T - C_T]\}.$$

- From the **viewpoint of the insurer** with objective $V(\cdot)$ and initial wealth ω the ask price, p^a, follows from

$$\sup_{X_T \in A(\omega + p^a)} \{V[X_T - C_T]\} = \sup_{X_T \in A(\omega)} \{V[X_T]\}.$$

- In general computing explicitly p^b and p^a is not in reach.
- (Market Consistency) If C_T is hedgeable, then

$$p^b = p^a = \mathbb{E}_P[\xi_T C_T] = e^{-rT} E_Q[C_T].$$

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Lower bound

- Assuming that decision makers are risk averse,

Theorem

Using the abusive notation $p^\bullet$ to reflect both p^a and p^b,

$$p^\bullet \geq \mathbb{E}_P[\xi_T.C_T] = e^{-rT}\mathbb{E}_Q[C_T].$$

Furthermore, the lower bound $\mathbb{E}_P[\xi_T.C_T]$ is the market price of the financial payoff $\mathbb{E}_P[C_T|\xi_T]$

- Note that

$$p^\bullet \geq e^{-rT}.\mathbb{E}_P[C_T] + Cov[C_T, \xi_T].$$

Comments

• Hence when the claim C_T and the state-price ξ_T are **negatively** correlated we find that $e^{-rT}.\mathbb{E}_P[C_T]$ **is no longer a lower bound** for p^b and p^a which contrasts with traditional bound stated in many actuarial textbooks on insurance pricing.

• Finally, remark that the inequality essentially states that both the insured and the insurer are prepared to agree on a price for the **insurance payoff** C_T which is larger than the price "as if C_T would be a **financial payoff"**.

Comments (Cont'd): 3 cases:

- **C_T is independent of the market**,

$$p^{\bullet} \geq e^{-rT}.\mathbb{E}_P[C_T].$$

- **C_T is positively correlated with the state-price process**, the classical lower bound $e^{-rT}\mathbb{E}_P[C_T]$ is now strictly improved.

$$p^{\bullet} \geq e^{-rT}.\mathbb{E}_P[C_T] + Cov[C_T, \xi_T] > e^{-rT}.\mathbb{E}_P[C_T].$$

- **C_T is negatively correlated with the state-price process**, the lower bound is smaller

$$p^{\bullet} \geq e^{-rT}.\mathbb{E}_P[C_T] + Cov[C_T, \xi_T].$$

If $C_T = S_T$, then $p^{\bullet} = S_0$ (market consistency) and $S_0 < e^{-rT}\mathbb{E}_P[S_T] = S_0 e^{(\mu-r)T}$

$$\begin{aligned} Cov(S_T, \xi_T) &= e^{-rT}(\mathbb{E}_Q[S_T] - \mathbb{E}_P[S_T]), \\ &= e^{-rT}(S_0 e^{rT} - S_0 e^{\mu T}), \end{aligned}$$

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Index-Linked Contract

- A life insurance company wants to reinsure payments of $(K-S_T)^+$ paid to a policyholder if alive at time T.

$$C_T = (K - S_T)^+ \mathbb{1}_{\tau > T}$$

where τ denotes the policyholder's time of death.

- Assume a Black Scholes financial market
- A reinsurer offers full coverage.

$$\mathbb{E}_P[\xi_T \mathbb{E}_P[C_T | \xi_T]] = \mathbb{E}_P[\xi_T C_T] = p(e^{-rT}K - S_0 + C_{bs}(S_0, K, T))$$

where $p = \mathbb{P}(\tau > T)$ and $C_{bs}(S_0, K, T)$ is the Black Scholes call price.

Illustration

Assume that u: insurer's utility

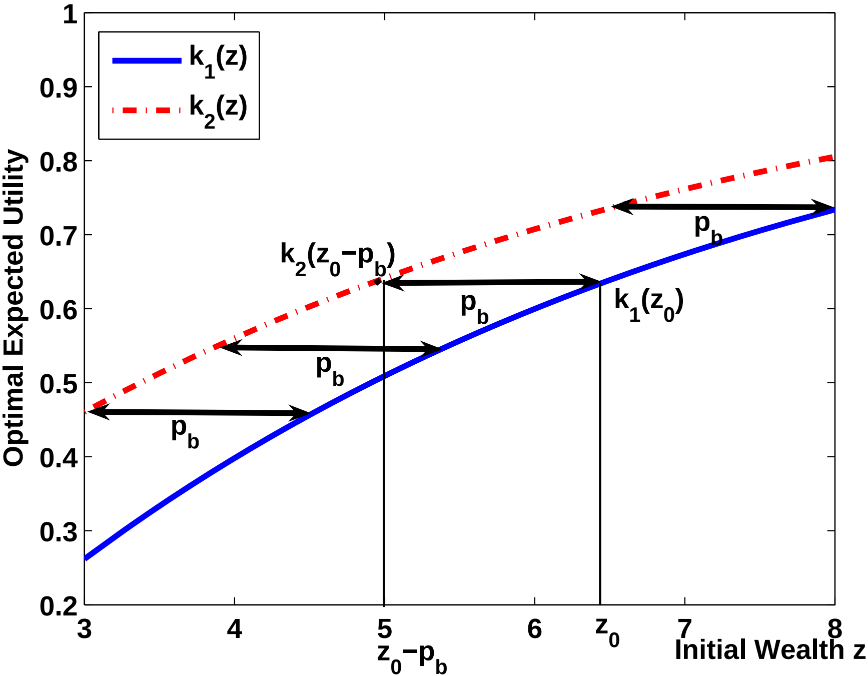
$$u(x) = 1 - \frac{\exp(-\gamma x)}{\gamma}.$$

where the absolute risk aversions $\gamma = 0.2$. Other parameters are $r = 2\%$, $\sigma = 0.2$, $\mu = 4\%$, $S_0 = 10$, $T = 1$, $K = 12$, $p = 0.7$. Next slide illustrates how to calculate bid prices where for a given wealth z

$$k_1(z) = \sup_{X_T \in A(z)} \mathbb{E}_P \left[u \left(X_T - C_T \right) \right]$$

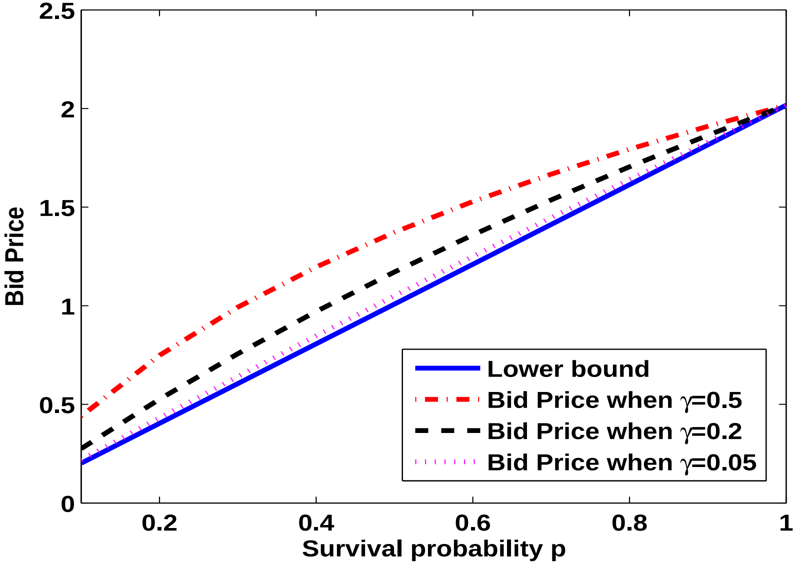
and

$$k_2(z) = \sup_{X_T \in A(z)} \mathbb{E}_P \left[u \left(X_T \right) \right].$$



k1(z)
k2(z)
Optimal Expected Utility
Initial Wealth z
k2(z0−pb)
k1(z0)
pb
pb
pb
pb
pb
z0−pb
z0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
3
4
5
6
7
8

Bid and ask prices with respect to survival probability p



Conclusion

- Preference-free bounds on market-consistent prices of financial and insurance claims
- These bounds correspond to prices of some financial payoffs that we give explicitly
- These bounds are robust in the sense that they are derived under rather mild assumptions
- Another lower bound can be found in the paper: it is derived under weaker assumptions on risk aversion

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Thanks!