Multivariate Option Pricing Using Copulae

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Introduction

Multivariate Options

- Financial products linked to more than one underlying
- Most are over-the-counter
- Some are listed on the New York Stock Exchange.

Introduction

Multivariate Options Pricing

- Multivariate Black and Scholes model.
- Stochastic correlation model (Galichon (2006), Langnau (2009)).
- Non-parametric estimation of the marginal risk neutral densities and of the risk neutral copula (Rosenberg (2000), Cherubini and Luciano (2002)).
- Parametric approach of dynamic copula modelling with GARCH(1,1) processes (Van den Goorbergh, Genest and Werker (2005)).

Underlying Indices Modeling

- Daily returns
  - $S_i(t)$: closing price of index i for the trading day t
  - $r_{i,t+1} = \log\left(S_i(t+1)/S_i(t)\right)$
- GARCH(1,1)
  - $$\begin{cases} r_{i,t+1} = \mu_i + \eta_{i,t+1}, \\ \sigma^2_{i,t+1} = w_i + \beta_i \sigma^2_{i,t} + \alpha_i (r_{i,t+1} - \mu_i)^2, \\ \eta_{i,t+1} | \mathcal{F}_t \sim_P \mathrm{N}(0, \sigma^2_{\mathrm{i,t}}) \end{cases}$$

    where $w_i > 0, \beta_i > 0$ and $\alpha_i > 0$
  - Standardized innovations for 3 indices (for example)

$$(Z_{1,s}, Z_{2,s}, Z_{3,s})_{s \leqslant t} := \left(\frac{\eta_{1,s}}{\sigma_{1,s}}, \frac{\eta_{2,s}}{\sigma_{2,s}}, \frac{\eta_{3,s}}{\sigma_{3,s}} \right)$$

Risk-Neutral Dynamics for each Index

Following Duan (1995), the log-returns under the risk neutral probability measure Q are given as follows

$$\begin{cases} r_{i,t+1} = r_f - \frac{1}{2}\sigma_{i,t}^2 + \eta_{i,t+1}^*, \\ \sigma_{i,t+1}^2 = w_i + \beta_i \sigma_{i,t}^2 + \alpha_i (r_{i,t+1} - \mu_i)^2, \\ \eta_{i,t+1}^* | \mathcal{F}_t \ \sim_Q \ \mathrm{N}(0, \sigma_{\mathrm{i,t}}^2) \end{cases}$$

where r_f is the (constant) daily risk-free rate on the market.

Pricing formula

$$\text{Initial Price} = e^{-r_f T} E_Q \left[g(S_1(T), S_2(T), S_3(T)) \right],$$

where

- T denotes the number of days between the issuance date and the maturity of the option.
- r_f is the risk-free rate.

$\Rightarrow$ **We need to understand the dependence under Q between S_1, S_2 and S_3.**

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Dependence Structure

- Need for a multivariate distribution with more than 2 dimensions: there are many bivariate copulae but a limited number of multivariate copulae
- Use of pair-copula construction (Aas, Czado, Frigessi and Bakken (2009) and Czado (2010))
- This method involves only bivariate copulae
- Example with 3 dimensions

Pair-copula Construction

- Joint density $f(x_1, x_2, x_3)$
- A possible decomposition by conditioning

$$f(x_1, x_2, x_3) = f(x_1|x_2, x_3) \times f_{2|3}(x_2|x_3) \times f_3(x_3).$$

- By Sklar's theorem

$$f(x_2, x_3) = c_{23}(F_2(x_2), F_3(x_3))f_2(x_2)f_3(x_3)$$

therefore

$$f_{2|3}(x_2|x_3) = c_{23}(F_2(x_2), F_3(x_3))f_2(x_2).$$

- Similarly we have

$$f_{1|2}(x_1|x_2) = c_{12}(F_1(x_1), F_2(x_2))f_1(x_1).$$

Pair-copula Construction

By Sklar's theorem for the conditional bivariate density

$$f(x_1, x_3|x_2) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))f_{1|2}(x_1|x_2)f_{3|2}(x_3|x_2)$$

and therefore

$$f(x_1|x_2, x_3) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))f_{1|2}(x_1|x_2).$$

It follows that

$$\begin{aligned} f(x_1, x_2, x_3) &= c_{12}(F_1(x_1), F_2(x_2))c_{23}(F_2(x_2), F_3(x_3)) \\ &\times c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))f_1(x_1)f_2(x_2)f_3(x_3). \end{aligned}$$

The corresponding copula density is therefore given by

$$c_{123}(u_1, u_2, u_3) = c_{12}(u_1, u_2)c_{23}(u_2, u_3).c_{13|2}(F_{1|2}(u_1|u_2), F_{3|2}(u_3|u_2))$$

It is called a D-vine in three dimensions and involves only bivariate copulae.

Example

IIL: "Capital Protected Notes Based on the Value of a Basket of Three Indices", issued by Morgan Stanley. The notes *IIL* are linked to

- S_1: the Dow Jones EURO STOXX 50^{SM} Index,
- S_2: the S&P 500 Index,
- S_3: the Nikkei 225 Index

Issue date: July 31st, 2006. Maturity date: July 20, 2010.
Initial price \$10.
Their final payoff is given by

$$\$10 + \$10 \max\left(\frac{m_1 S_1(T) + m_2 S_2(T) + m_3 S_3(T) - 10}{10}, 0\right)$$

where $m_i = \frac{10}{3S_i(0)}$ such that $m_1 S_1(0) + m_2 S_2(0) + m_3 S_3(0) = 10$ and the % weighting in the basket is 33.33% for each index.

GARCH(1,1) parameters

The table next slide:

Estimated parameters of GARCH(1,1)

- 3 indices:
  - **the STOXX50**,
  - **the S&P500**,
  - **the NIK225**.
- $\bar{\sigma}_{i,t}$ denotes the average of the daily volatilities over the period under study (full period is July 2006 to November 2009).

The table highlights different regimes of the economy (time varying parameters for the GARCH(1,1) model) and changes in volatility.

	Full sample	period 1	period 2	period 3
$\hat{\mu}_1$	0.000414	0.000664	-0.000513	0.000593
$\hat{\omega}_1$	1.85e-06	2.72e-06	8.95e-06	5.42e-06
$\hat{\alpha}_1$	0.0932	0.0338	0.0513	0.119
$\hat{\beta}_1$	0.900	0.903	0.899	0.876
$\hat{\mu}_2$	0.000350	0.000907	-0.000494	0.000743
$\hat{\omega}_2$	3.76e-06	9.88e-06	1.027e-05	7.57e-06
$\hat{\alpha}_2$	0.1343	0.1598	0.1482	0.1062
$\hat{\beta}_2$	0.8575	0.7275	0.8063	0.8854
$\hat{\mu}_3$	0.000107	0.000525	-0.000594	0.0000213
$\hat{\omega}_3$	4.63e-06	4.75e-06	6.09e-06	1.83e-05
$\hat{\alpha}_3$	0.127	0.0643	0.142	0.197
$\hat{\beta}_3$	0.863	0.896	0.851	0.782
$\bar{\sigma}_{1,t}\sqrt{250}$	24.8%	14.5%	21.2%	38.7%
$\bar{\sigma}_{2,t}\sqrt{250}$	23.2%	10.3%	20.6%	38.8%
$\bar{\sigma}_{3,t}\sqrt{250}$	27.4%	17.5%	25.8%	39.1%

Methodology

- Example with 3 indices
  1. Identify the 2 couples that have the most dependence.
  2. Identify the family of copula using empirical contour plots and Cramer von Mises Goodness of Fit test.
  3. Generate the conditional data and identify the copula of the conditional data.
- Illustration with the contract IIL

Note that the Pair-Copula Construction depends on the order of the indices (item 1 is arbitrary).

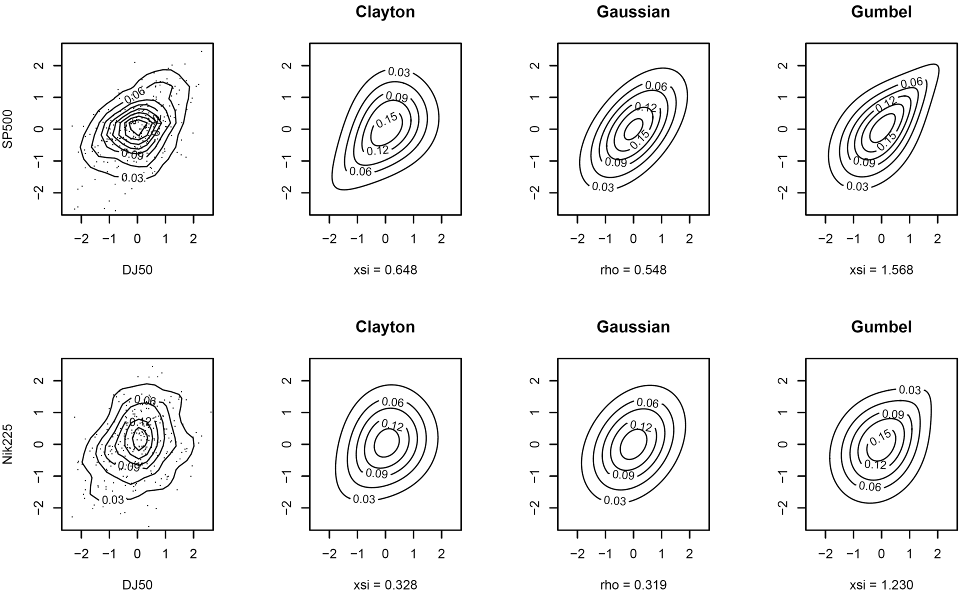
	$S_1 - S_2$	$S_1 - S_3$	$S_2 - S_3$
Full	0.404	0.202	0.079
Period 1	0.314	0.197	0.104
Period 2	0.384	0.239	0.075
Period 3	0.495	0.181	0.062

Overall dependence measured by the Kendall's Tau for the full sample and then for each of the 3 periods

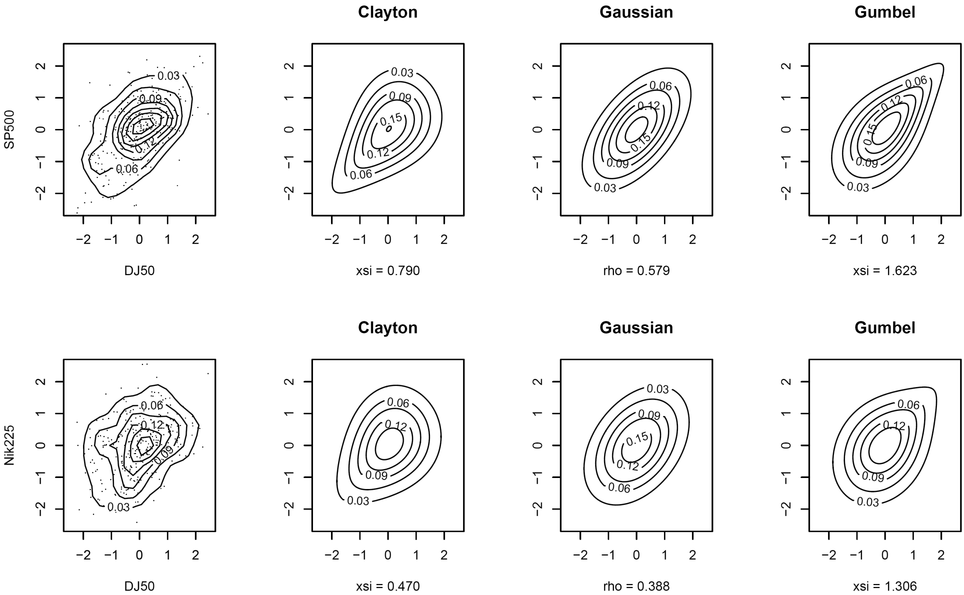
Contour plots

- We draw the contour plots for $S_1 - S_2$ and $S_1 - S_3$
- The empirical contours are compared with theoretical contours.
- All parameter estimates are obtained by maximum likelihood estimation.

We only present the 1st and 2nd period.



Clayton
Gaussian
Gumbel
SP500
DJ50
xsi = 0.648
rho = 0.548
xsi = 1.568
Clayton
Gaussian
Gumbel
Nik225
DJ50
xsi = 0.328
rho = 0.319
xsi = 1.230



Clayton
Gaussian
Gumbel
SP500
DJ50
xsi = 0.790
rho = 0.579
xsi = 1.623
Clayton
Gaussian
Gumbel
Nik225
DJ50
xsi = 0.470
rho = 0.388
xsi = 1.306

Conditional copula

We compute the copula

$$C_{23|1}$$

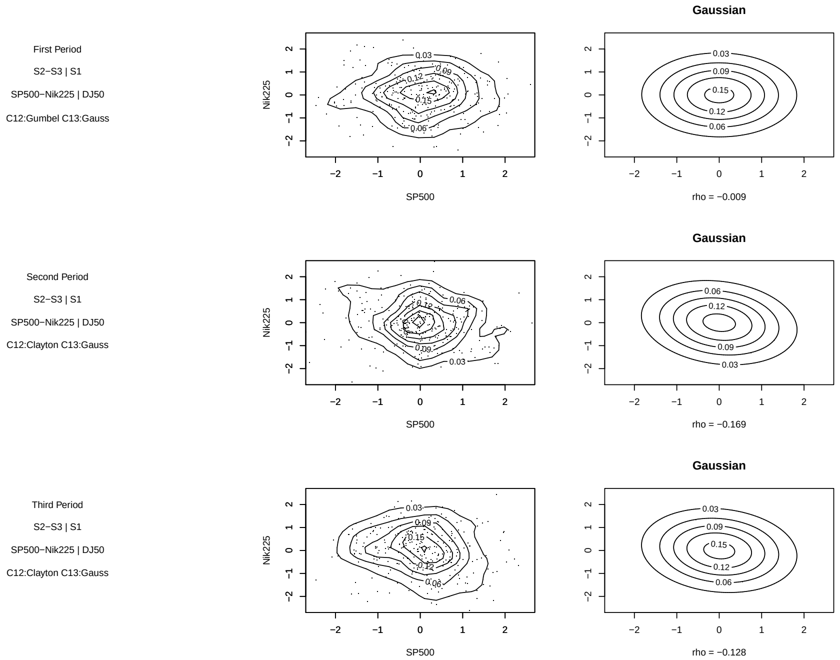
between the conditional distributions of S_2 given S_1 and S_3 given S_1 as follows

$$\begin{aligned} u_{2|1s} &= F_{2|1,\theta_{12}}(u_{2s}|u_{1s},\widehat{\theta}_{12}^{P}) \\ u_{3|1s} &= F_{3|1,\theta_{13}}(u_{3s}|u_{1s},\widehat{\theta}_{13}^{P}) \end{aligned}$$

where the conditional distribution $F_{2|1,\theta_{12}^{P}}$ is obtained by

$$F(u_2|u_1,\widehat{\theta}_{12}^{P}) = \frac{\partial}{\partial u_1} C_{12}(u_2|u_1,\widehat{\theta}_{12}^{P}) =: h(u_2,u_1,\widehat{\theta}_{12}^{P})$$

and $F_{3|1,\theta_{13}^{P}}$ similarly.
(For the second period, we assume a Clayton copula between S_1 and S_2 and a Gaussian copula between S_1 and S_3.)


First Period
S2−S3 | S1
SP500−Nik225 | DJ50
C12:Gumbel C13:Gauss
Gaussian
Nik225
SP500
rho = −0.009
Second Period
S2−S3 | S1
SP500−Nik225 | DJ50
C12:Clayton C13:Gauss
Gaussian
Nik225
SP500
rho = −0.169
Third Period
S2−S3 | S1
SP500−Nik225 | DJ50
C12:Clayton C13:Gauss
Gaussian
Nik225
SP500
rho = −0.128

Comments on the last figure

- For each subperiod the dependence for the conditional distribution is very weak, it could even be slightly negative.
- Note that the Gumbel and Clayton copulas cannot be used to model negative dependence.
- S_1 and S_3 were weakly dependent. Conditionally to S_2, they look independent.
- This suggests that the dependence between the Asian index and the US index is fully captured by the European index.

The remaining question is about the choice of the parameter set Θ^Q. One needs past observations of prices of the trivariate option, say at dates t_i, $i = 1..n$, the set of parameters Θ_Q needed to characterize the copula C^Q is calculated at time t such that it minimizes the sum of quadratic errors

$$\min_{\Theta_Q} \sum_{i=1}^{n} \left(\hat{g}_{t_i}^{mc}(\Theta_Q) - g_{t_i}^{M} \right)^2 .$$

where

- $g_{t_i}^{M}$ denotes the market price of the trivariate option observed in the market at the date t_i,
- $\hat{g}_{t_i}^{mc}$ is the Monte Carlo estimate of its price obtained by the procedure described previously.

The contract IIL was issued at the price \$10.
The two indices that are mostly dependent are S_1 and S_2, we look at the sensitivity of the price of the contract with respect to the choice of the copula and its parameter to model the dependence between S_1 and S_2.
We observe that:

- The contract is more expensive when the copula is Clayton or Gumbel rather than Gauss. Therefore the choice of the copula family is important.
- In general the parameter of the copula under Q, such that the market price of the contract is equal to the model price, is different from the parameter of the copula estimated under P
- This would suggest that the copula under P may be different than the copula under Q

Remarks

- Prices are very sensitive to the risk-free rate r_f. How to choose it?
  - Use options written on **only one index** and find the risk-free rate such that the price of the option with the GARCH(1,1) model and Duan's transformation matches the market price.
  - We found that the risk-free rate such that the model price is equal to the market price is approximately the US zero-coupon yield curve.(This is good!)
  - This last observation shows that the GARCH(1,1) model is a good model to price contracts linked to one index.
- Other observations
  - This is a preliminary study
  - Need to investigate the sensitivity to parameters of GARCH(1,1)
  - Need to study other contracts

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Conclusions

- The paper proposes a methodology to price multivariate derivatives
  1. use of GARCH(1,1) to model underlying indices
  2. use of Pair Copula Construction
- Through Monte Carlo simulations, we show that the dependency structure has an important impact on the price of multivariate derivatives.
- This model is accurate for unidimensional derivatives. Prices are sensible.
- To fit multivariate derivatives prices, one needs to adjust parameters of the historical copula. The risk-neutral copula may be different from the historical copula.
- This discrepancy may also come from other factors such that a higher margin from issuers. It may also be due to the fact that the illiquidity of the secondary market for retail products.
- Further tests are needed with more data to draw firmer conclusions.