Optimal Portfolio
Under Worst-Case Scenarios

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Outline of the talk

1. The Growth Optimal Portfolio and Issues with Traditional Diversification Strategies
2. Characterization of optimal investment strategies for an investor with **law-invariant preferences** and a **fixed investment horizon**
3. Issues with these optimal strategies
4. Extension to the case when investors have **state-dependent constraints**.

Contributions

1. A better understanding of the link between Growth Optimal Portfolio and optimal investment strategies
2. Understanding how **lowest outcomes of optimal strategies always happen in the worse states of the economy**.
3. Develop **innovative** strategies to cope with this observation.
4. Implications in terms of **assessing the risk and return** of a strategy and in terms of **reducing systemic risk**

Part I:

Traditional

Diversification Strategies

Growth Optimal Portfolio (GOP)

- The **Growth Optimal Portfolio** (GOP) maximizes expected logarithmic utility from terminal wealth.
- It has the property that it **almost surely accumulates more wealth than any other strictly positive portfolios after a sufficiently long time**.
- Under general assumptions on the market, the GOP is a diversified portfolio.
- Details in Platen (2006).

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For example, in the Black-Scholes model

- A Black-Scholes financial market (mainly for ease of exposition)
- Risk-free asset $\{B_t = B_0 e^{rt},\ t \geqslant 0\}$
-

$$\begin{cases} \frac{dS_t^1}{S_t^1} = \mu_1 dt + \sigma_1 dW_t^1 \\ \frac{dS_t^2}{S_t^2} = \mu_2 dt + \sigma_2 dW_t \end{cases}, \tag{1}$$

where W^1 and W are two correlated Brownian motions under the physical probability measure $\mathbb{P}$.

$$W_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$$

where W^1 and W^2 are independent.

Growth Optimal Portfolio (GOP)

In the 2-dimensional Black-Scholes setting,

- The payoff of a constant-mix strategy is

$$S_t^\pi = S_0^\pi \ \exp(X_t^\pi)$$

  where X_t^π is normal.

- The GOP is a constant-mix strategy with $X_t^\pi = \left(\mu_\pi - \frac{1}{2}\sigma_\pi^2\right) t + \sigma_\pi W_t^\pi$, that **maximizes the expected growth rate** $\mu_\pi - \frac{1}{2}\sigma_\pi^2$. It is

$$\pi^\star = \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1}) \,. \tag{2}$$

  Denote by $S_{\pi^\star} = S^\star$ the GOP.

Market Crisis

The **growth optimal portfolio** $S^\star$ can also be interpreted as a major market index. Hence it is intuitive to define a stressed market (or crisis) at time T as an event where *the market* - materialized through $S^\star$ - **drops below its Value-at-Risk** at some high confidence level. The corresponding states of the economy verify

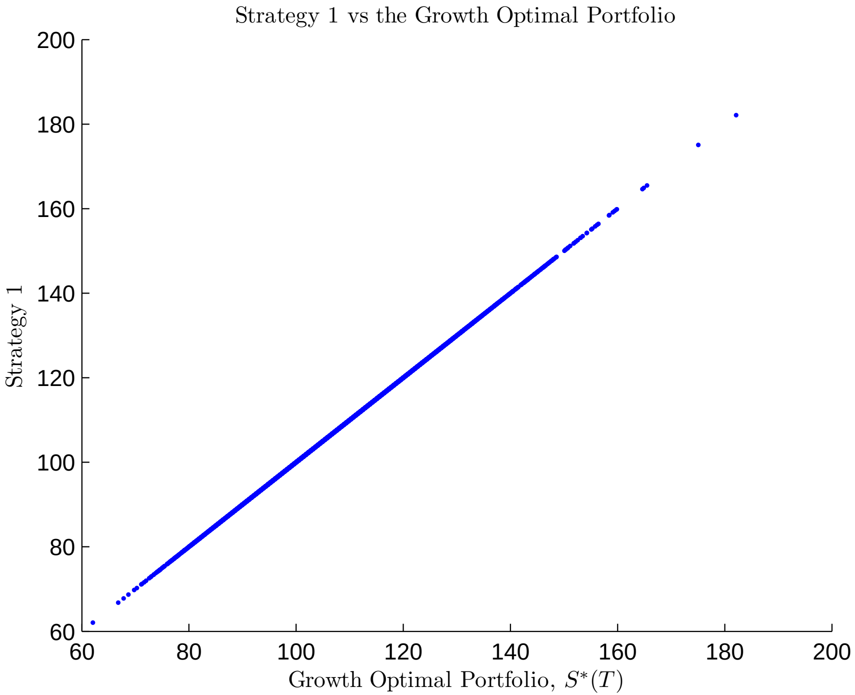
$$\textit{Crisis states} = \{S^\star_T < q_\alpha\}\,, \tag{3}$$

where q_α is such that $P(S^\star_T < q_\alpha) = 1 - \alpha$ and α is typically high (e.g. $\alpha = 0.98$).

Srategy 1: GOP

We invest fully in the GOP.

In a crisis (GOP is low), our portfolio is low!


Strategy 1 vs the Growth Optimal Portfolio
Strategy 1
Growth Optimal Portfolio, $S^*(T)$
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Srategy 2: Buy-and-Hold

The buy-and-hold strategy is the simplest investment strategy. An initial amount V_0 is used to purchase w_0 units of the bank account and w_i units of stock S^i ($i = 1, 2$) such that

$$V_0 = w_0 + w_1 \; S_0^1 + w_2 \; S_0^2,$$

and no further action is undertaken.

Example with 1/3 invested in each asset (bank, S_1 and S_2) on next slide.

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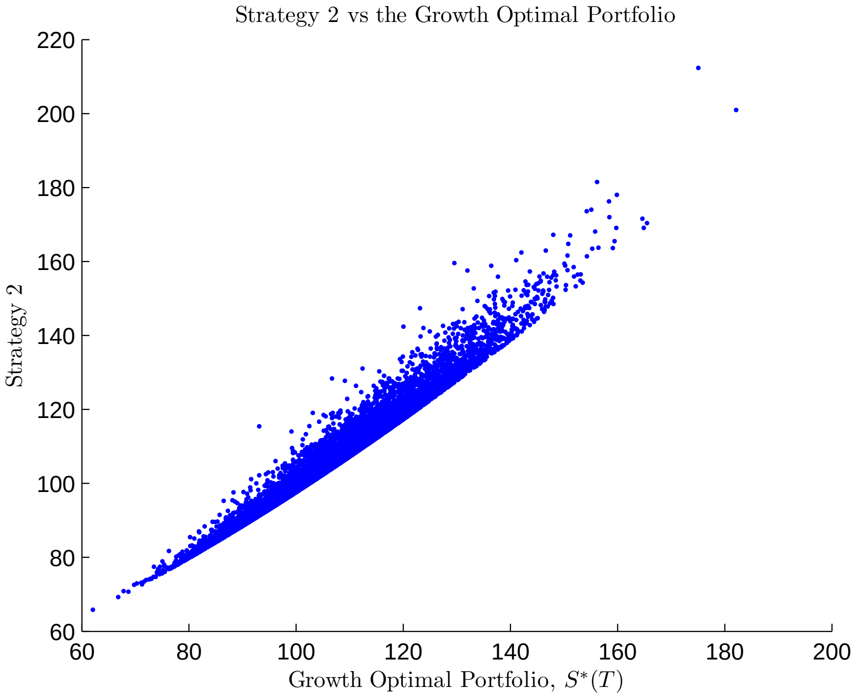

Strategy 2 vs the Growth Optimal Portfolio
Strategy 2
Growth Optimal Portfolio, $S^*(T)$
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Strategy 3: Constant-Mix Strategy

- dynamic rebalancing to preserve the initial target allocation
- For an initial investment V_0, V_T is given by

$$V_T = V_0 \frac{S_T^{\pi}}{S_0^{\pi}},$$

  where π is the vector of proportions.
- constant-mix portfolios given by $\pi = \alpha\pi^{\star}$ with $\alpha > 0$ and where $\pi^{\star}$ is the optimal proportion for the GOP, are optimal strategies for CRRA expected utility maximizers. Precisely, CRRA investors with a constant relative risk aversion coefficient $\eta > 0$ have utility

$$U(x) = \begin{cases} \frac{x^{1-\eta}}{1-\eta} & \text{when} \quad \eta \neq 1 \\ \log(x) & \text{when} \quad \eta = 1, \end{cases}$$

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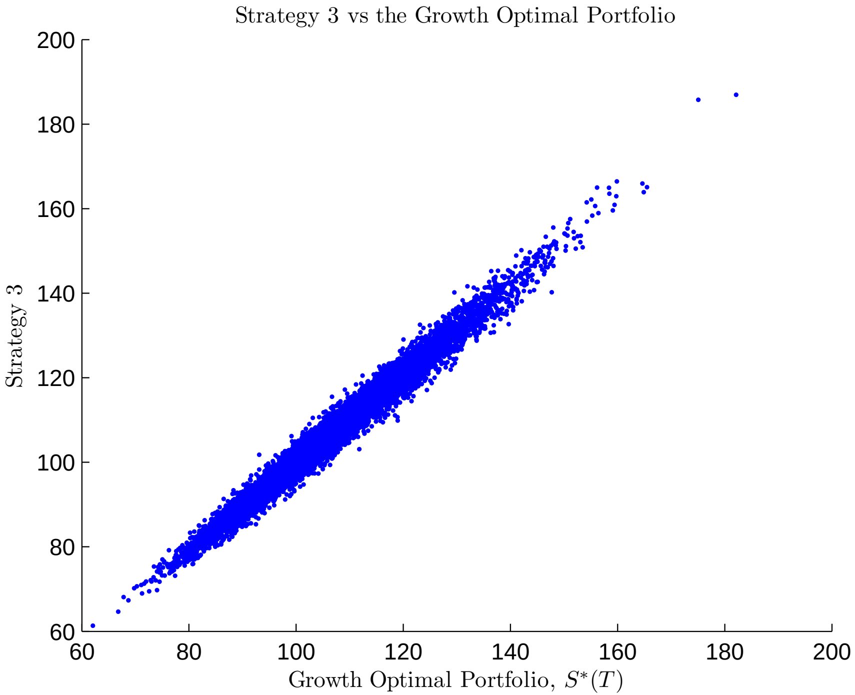
Strategy 3 vs the Growth Optimal Portfolio
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- These three traditional diversification strategies do not offer protection during a crisis.
- In a more general setting, optimal strategies share the same problem...

Part II:

Optimal portfolio selection

for law-invariant preferences

Stochastic Discount Factor and Real-World Pricing:

The GOP can be used as numeraire to price under P

$$\left\{ \begin{array}{c} \textit{Price of} \\ X_T \textit{ at } 0 \end{array} \right\} = E_Q[e^{-rT}X_T] = E_P[\xi_T X_T] = E_P\left[\frac{X_T}{S_T^\star}\right]$$

where $S_0^\star = 1$.

Optimal Portfolio Selection Problem

$$\max_{X_T} \mathcal{U}(X_T)$$

subject to a given "cost of X_T" (equal to initial wealth)

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Cost-efficient strategies (Dybvig (1988))

Consider an investor with **fixed investment horizon** and objective function to optimize $\mathcal{U}(\cdot)$.

- **Law-invariant** preferences

$$X_T \sim Y_T \Rightarrow \mathcal{U}(X_T) = \mathcal{U}(Y_T)$$

- **Increasing** preferences

$$X_T \sim F, Y_T \sim G, \forall x, F(x) \leqslant G(x) \Rightarrow \mathcal{U}(X_T) \geqslant \mathcal{U}(Y_T)$$

A strategy (or a payoff) is cost-efficient

if any other strategy that generates the same distribution under P costs at least as much.

The optimal strategy for $\mathcal{U}$ must be **cost-efficient**.

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Optimal Portfolio and Cost-efficiency

Consider an investor with **increasing law-invariant** preferences and a **fixed** horizon. Denote by X_T the investor's final wealth. The optimal strategy solves a cost-efficiency problem

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\frac{X_T}{S_T^\star}\right]$$

Reciprocally a cost-efficient strategy with a continuous distribution F corresponds to the optimum of an expected utility investor for

$$U(x) = \int_0^x G^{-1}(1 - F(y))dy$$

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Sufficient Condition for Cost-efficiency

A random pair (X, Y) is comonotonic if

there exists a non-decreasing relationship between them.

Theorem (Sufficient condition for cost-efficiency)

Any random payoff X_T with the property that $(X_T, S_T^\star)$ is **comonotonic** *is* **cost-efficient**.

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Idea of the proof

$$\min_{X_T} \mathbb{E}\left[\frac{X_T}{S_T^\star}\right]$$
$$\text{subject to} \begin{cases} X_T \sim F \\ \frac{1}{S_T^\star} \sim G \end{cases}$$

Recall that

$$\text{corr}\left(X_T, \frac{1}{S_T^\star}\right) = \frac{\mathbb{E}\left[X_T \frac{1}{S_T^\star}\right] - \mathbb{E}[\frac{1}{S_T^\star}]\mathbb{E}[X_T]}{\text{std}(\frac{1}{S_T^\star})\,\text{std}(X_T)}.$$

We can prove that when the distributions for both X_T and $\frac{1}{S_T^\star}$ are fixed, we have

$$(\mathbf{X_T}, \mathbf{S_T^\star}) \text{ is comonotonic} \Rightarrow \text{corr}\left[\mathbf{X_T}, \frac{\mathbf{1}}{\mathbf{S_T^\star}}\right] \text{ is minimal.}$$

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Explicit Representation for Cost-efficiency

Theorem

Consider the following optimization problem:

$$PD(F) := \min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\frac{X_T}{S_T^\star}\right]$$

Assume $S_T^\star$ is continuously distributed, *then the optimal strategy is*

$$X_T^\star = F^{-1}\left(F_{S_T^\star}(S_T^\star)\right).$$

Note that $X_T^\star \sim F$ and $X_T^\star$ is a.s. **unique** *such that* $PD(F) = c(X_T^\star)$

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Black-Scholes Model

To be cost-efficient, the contract has to be a European derivative written on $S_T^\star$ and non-decreasing w.r.t. $S_T^\star$. In this case,

$$X_T^\star = F^{-1}\left(F_{S_T^\star}\left(S_T^\star\right)\right)$$

Corollary

Path-dependent derivatives are always inefficient in the Black-Scholes framework.

Part III:

Investment under Worst-Case Scenarios

Investment with State-Dependent Constraints

Problem considered so far

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\frac{X_T}{S_T^\star}\right].$$

A payoff that solves this problem is **cost-efficient**.

New Problem

$$\min_{\{V_T \mid V_T \sim F,\, \mathbb{S}\}} \mathbb{E}\left[\frac{V_T}{S_T^\star}\right].$$

where $\mathbb{S}$ denotes a set of constraints. A payoff that solves this problem is called a $\mathbb{S}-$**constrained cost-efficient payoff**.

Type of Constraints

We are able to find optimal strategies with final payoff V_T

- with an additional probability constraint

$$P(S_T^\star \leqslant s, V_T \leqslant v) = \beta$$

- with a set of probability constraints

$$\forall (s,v) \in \mathbb{S},\ P(S_T^\star \leqslant s, V_T \leqslant v) = Q(s,v)$$

  where Q is an appropriate given function and $\mathbb{S}$ verifies some properties.

- in particular, assuming that the final payoff of the strategy is independent of $S_T^\star$ during a crisis (defined as $S_T^\star \leqslant q_\alpha$),

$$\forall s \leqslant q_\alpha, v \in \mathbb{R}, P(S_T^\star \leqslant s, V_T \leqslant v) = P(S_T^\star \leqslant s)P(V_T \leqslant v)$$

Independence in the Tail - Strategy 4: Path-dependent

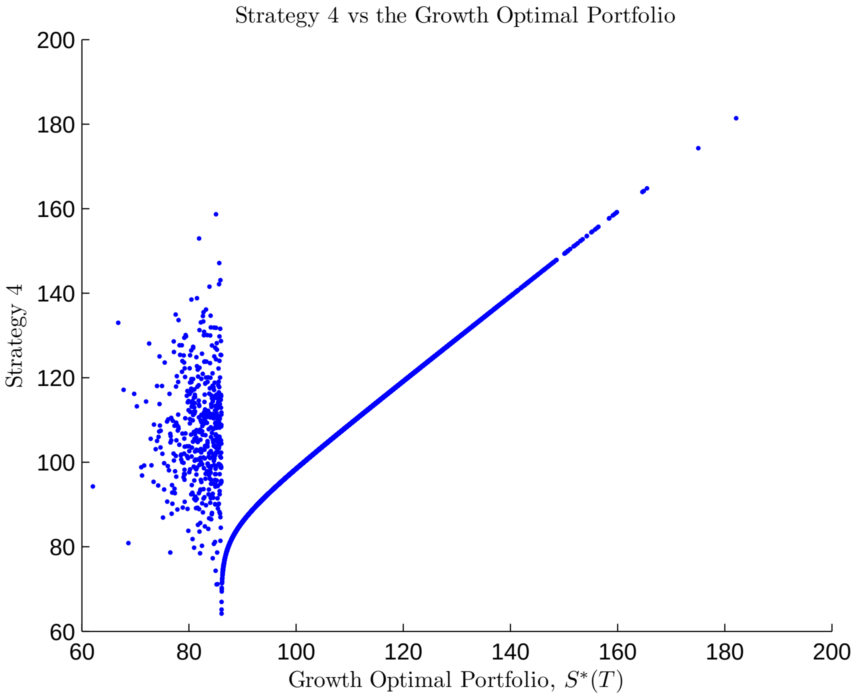
Theorem

The cheapest path-dependent strategy with a cumulative distribution F but such that it is independent of $S_T^\star$ when $S_T^\star \leqslant q_\alpha$ can be constructed as

$$V_T^\star = \begin{cases} F^{-1}\left(\frac{F_{S_T^\star}(S_T^\star)-\alpha}{1-\alpha}\right) & \text{when} \quad S_T^\star > q_\alpha, \\ F^{-1}\left(\Phi\left(\frac{\ln\left(\frac{S_t^\star}{\left(S_T^\star\right)^{t/T}}\right)-(1-\frac{t}{T})\ln(S_0^\star)}{\sigma_\star\sqrt{t-\frac{t^2}{T}}}\right)\right) & \text{when} \quad S_T^\star \leqslant q_\alpha, \end{cases} \tag{4}$$

where $t \in (0, T)$ can be chosen freely.

(**No uniqueness and path-independence anymore**).



Strategy 4 vs the Growth Optimal Portfolio
Strategy 4
Growth Optimal Portfolio, $S^*(T)$
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Independence in the Tail - Strategy 5: Path-independent

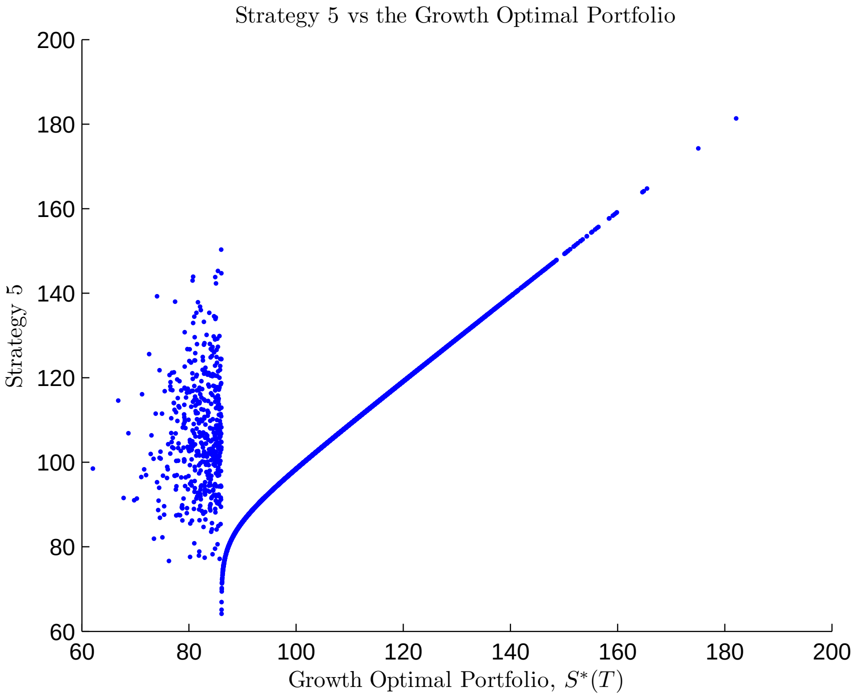
In a financial market that **contains at least two assets** that are continuously distributed, the **cheapest** path-independent strategy with a cumulative distribution F but such that it is **independent** of $S_T^\star$ when $S_T^\star \leqslant q_\alpha$ can be constructed as

$$Z_T^\star = \begin{cases} F^{-1}\left(\frac{F_{S_T^\star}(S_T^\star)-\alpha}{1-\alpha}\right) & \text{when} \quad S_T^\star > q_\alpha \\ F^{-1}(\Phi(A)) & \text{when} \quad S_T^\star \leqslant q_\alpha \end{cases}. \qquad (5)$$

where A is given as

$$A = \Phi\left(\frac{\frac{1}{\sigma_1}\left[\ln\left(\frac{S_T^1}{S_0^1}\right) - \left(\mu_1 - \frac{\sigma_1^2}{2}\right)T\right] - \frac{\rho_\star}{\sigma_\star}\left[\ln\left(\frac{S_T^\star}{S_0^\star}\right) - \left(\mu_\star - \frac{\sigma_\star^2}{2}\right)T\right]}{\sqrt{(1-\rho_\star^2)T}}\right),$$

with $\rho_\star = \frac{\pi_1^\star \sigma_1^2 + \sigma_1 \sum_{i=2}^n \pi_i^\star \sigma_i \rho_{1i}}{\sigma_1 \sigma_\star}$ and $\pi_i^\star$ denotes the $i-$th element of the growth optimal portfolio $\pi_\star$.


Strategy 5 vs the Growth Optimal Portfolio
Strategy 5
Growth Optimal Portfolio, $S^*(T)$
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Part IV:

Investment under

Worst-Case Scenarios

Some numerical examples

Copulas and Sklar's theorem

The joint cdf of a couple $(S_T^\star, X)$ can be decomposed into 3 elements

- The marginal cdf of $S_T^\star$: H
- The marginal cdf of X_T: F
- A copula C

such that

$$P(S_T^\star \leqslant s, X_T \leqslant x) = C(H(s), F(x))$$

Other Types of Dependence

Independence in the tail:

$$\forall s \leqslant q_\alpha, v \in \mathbb{R}, P(S_T^\star \leqslant s, V_T \leqslant v) = P(S_T^\star \leqslant s)P(V_T \leqslant v)$$

This corresponds to the independence copula $C(u, v) = uv$

- We were also able to derive formulas for optimal strategies that generate a Gaussian distribution in the tail with a correlation coefficient of -0.5.
- Similarly for Clayton or Frank dependence.

Optimal Investment with a Clayton Tail Dependence

Let $Q^\star$ be the **Clayton** copula $Q^\star(u,v) = (u^{-a} + v^{-a} - 1)^{-1/a}$.
Assume that the payoff V_T of the strategy has a cdf F and has to satisfy the following dependency when the market is in a crisis

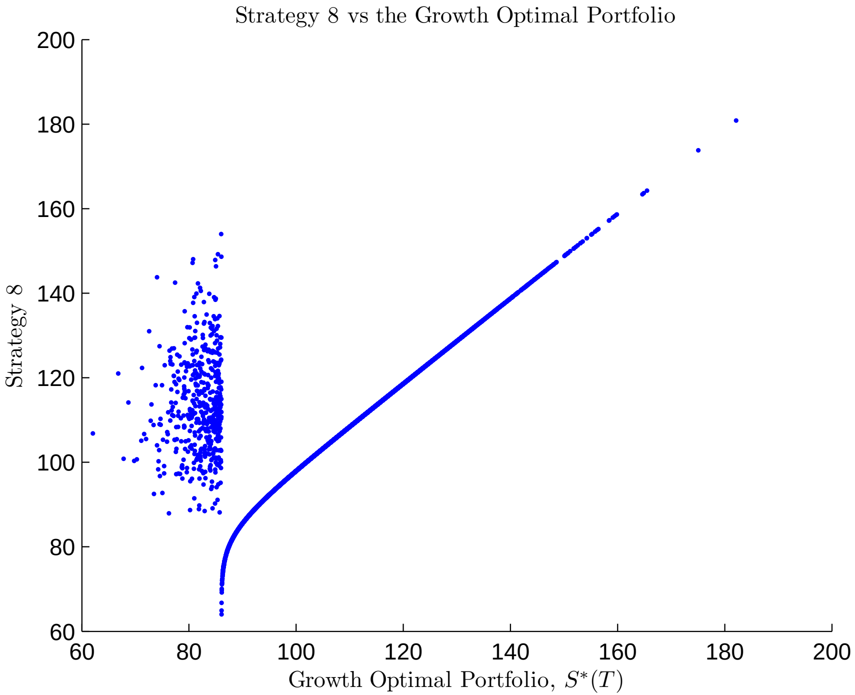
$$\forall s \in [0, q_\alpha],\ y \in \mathbb{R},\ P(S_T^\star \leqslant s,\ V_T \leqslant y) = F(y) - Q^\star(1 - F_{S_T^\star}(s), F(y))$$

Then the **cheapest** strategy $V_T^\star$ with cdf F that verifies this **Clayton** dependence in the tail is

$$V_T^\star = \begin{cases} F^{-1}\left(\left[(F_{S_T^\star}(S_T^\star) - \alpha)^{-a} - (1-\alpha)^{-a} + 1\right]^{-1/a}\right) & \text{if} \quad S_T^\star > q_\alpha \\ F^{-1}\left(g\left(1 - F_{S_T^\star}(S_T^\star), j_{F_{S_T^\star}(S_T^\star)}(F_{Z_T}(Z_T))\right)\right) & \text{if} \quad S_T^\star \leqslant q_\alpha, \end{cases}$$

where Z_T is such that $(S_T^\star, Z_T)$ is continuously distributed (with copula J) and where g is known explicitly:

$$g(u,v) = \left[u^{-a}\left(v^{-a/(1+a)} - 1\right) + 1\right]^{-1/a}.$$


Strategy 8 vs the Growth Optimal Portfolio
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Strategy 8
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Growth Optimal Portfolio, $S^*(T)$

Some numerical results

We define two events related to *the market*, i.e. the market crisis $C = \{S_T^\star < q_\alpha\}$ and a decrease in the market $D = \left\{S_T^\star < S_0^\star e^{rT}\right\}$. We further define two events for the portfolio value by $A = \left\{V_T < V_0 e^{rT}\right\}$ and $B = \left\{V_T < 75\% V_0 e^{rT}\right\}$

	T	Cost	Sharpe	$P(A\|C)$	$P(A\|D)$	$P(B\|C)$
GOP	5	100	0.266	1.00	1.00	1.00
Buy-and-Hold	5	100	0.239	0.9998	0.965	0.99
Independence	5	101.67	0.214	0.46	0.94	0.13
Gaussian	5	103.40	0.159	0.12	0.90	0.01
Clayton	5	102.35	0.193	0.24	0.91	0.02

Conclusions

- **Cost-efficiency:** a preference-free framework for ranking different investment strategies.
- **Characterization of optimal portfolio strategies** for investors with law invariant preferences and a fixed horizon.

► **Lowest outcomes in worst states** of the economy

- Optimal investment choice under **state-dependent** constraints.
  - not always non-decreasing with the GOP $S_T^\star$.
  - not anymore unique
  - could be path-dependent.

► **Trade-off** between losing "utility" and gaining from better fit of the investor's preferences.

More Implications

- The new strategies do not incur their biggest losses in the worst states in the economy.
- can be used to **reduce systemic risk**.
  - the idea of assessing risk and performance of a portfolio not only by looking at its final distribution but also by looking at its interaction with the economic conditions is indeed related to the increasing concern to evaluate systemic risk.
  - Acharya (2009) explains that regulators should "be regulating each bank as a function of both its joint (correlated) risk with other banks as well as its individual (bank-specific) risk".
  - An insight of this work is that if all institutional investors implement strategies that are resilient against crisis regimes, as we propose, then systemic risk can be diminished.

Do not hesitate to contact me to get updated working papers!

References

- Bernard, C., Boyle P., Vanduffel S., 2011, "Explicit Representation of Cost-efficient Strategies", available on SSRN.
- Bernard, C., Jiang, X., Vanduffel, S., 2012. "Note on Improved Frechet bounds and model-free pricing of multi-asset options", *Journal of Applied Probability*.
- Bernard, C., Maj, M., Vanduffel, S., 2011. "Improving the Design of Financial Products in a Multidimensional Black-Scholes Market,", *North American Actuarial Journal*.
- Bernard, C., Vanduffel, S., 2011. "Optimal Investment under Probability Constraints," *AfMath Proceedings*.
- Bernard, C., Vanduffel, S., 2012. "Financial Bounds for Insurance Prices," *Journal of Risk Insurance*.
- Cox, J.C., Leland, H., 1982. "On Dynamic Investment Strategies," *Proceedings of the seminar on the Analysis of Security Prices*,(published in 2000 in *JEDC*).
- Dybvig, P., 1988a. "Distributional Analysis of Portfolio Choice," *Journal of Business*.
- Dybvig, P., 1988b. "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market," *Review of Financial Studies*.
- Goldstein, D.G., Johnson, E.J., Sharpe, W.F., 2008. "Choosing Outcomes versus Choosing Products: Consumer-focused Retirement Investment Advice," *Journal of Consumer Research*.
- Jin, H., Zhou, X.Y., 2008. "Behavioral Portfolio Selection in Continuous Time," *Mathematical Finance*.
- Nelsen, R., 2006. *"An Introduction to Copulas"*, Second edition, Springer.
- Pelsser, A., Vorst, T., 1996. "Transaction Costs and Efficiency of Portfolio Strategies," *European Journal of Operational Research*.
- Platen, E., 2005. "A benchmark approach to quantitative finance," Springer finance.
- Tankov, P., 2011. "Improved Frechet bounds and model-free pricing of multi-asset options," *Journal of Applied Probability, forthcoming*.
- Vanduffel, S., Chernih, A., Maj, M., Schoutens, W. 2009. "On the Suboptimality of Path-dependent Pay-offs in Lévy markets", *Applied Mathematical Finance*.

Part V:

Proofs with Copulas

Optimal Portfolio under Tail Dependence

Copulas and Sklar's theorem

The joint cdf of a couple (ξ_T, X) can be decomposed into 3 elements

- The marginal cdf of ξ_T: G
- The marginal cdf of X_T: F
- A copula C

such that

$$P(\xi_T \leqslant \xi, X_T \leqslant x) = C(G(\xi), F(x))$$

Where do copulas appear?

in the derivation of "cost-efficient" strategies...

Solving the cost-efficiency problem amounts to finding bounds on copulas!

$$\min_{X_T} \; \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \;\; \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

Proof of the cost-efficient payoff

$$\min_{X_T} \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

The distribution G is known and depends on the financial market. Let C denote a copula for (ξ_T, X).

$$\mathbb{E}[\xi_T X] = \int\int (1 - G(\xi) - F(x) + C(G(\xi), F(x)))dxd\xi, \quad (6)$$

The lower bound for $\mathbb{E}[\xi_T X]$ is derived from the lower bound on C

$$\max(u + v - 1, 0) \leqslant C(u, v)$$

(where $\max(u + v - 1, 0)$ corresponds to the **anti-monotonic** copula).

$$E[\xi_T F^{-1}(1 - G(\xi_T))] \leqslant E[\xi_T X_T]$$

then $X_T^\star = F^{-1}(1 - G(\xi_T))$ has the minimum price for the cdf F.

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then $X_T^\star = F^{-1}\left(1 - G\left(\xi_T\right)\right)$ has the minimum price for the cdf F.

Sufficient condition for the existence

Theorem

Let $t \in (0, T)$. If there exists a copula L satisfying $\mathbb{S}$ such that $L \leqslant C$ (pointwise) for all other copulas C satisfying $\mathbb{S}$ then the payoff $Y_T^\star$ given by

$$Y_T^\star = F^{-1}(f(\xi_T, \xi_t))$$

is a $\mathbb{S}$-constrained cost-efficient payoff. Here $f(\xi_T, \xi_t)$ is given by

$$f(\xi_T, \xi_t) = \left(\ell_{G(\xi_T)}\right)^{-1}\left[j_{G(\xi_T)}(G(\xi_t))\right],$$

where the functions $j_u(v)$ and $\ell_u(v)$ are defined as the first partial derivative for $(u, v) \to J(u, v)$ and $(u, v) \to L(u, v)$ respectively and where J denotes the copula for the random pair (ξ_T, ξ_t).

If (U, V) has a copula L then $\ell_u(v) = \mathbb{P}(V \leqslant v | U = u)$.

When $\mathbb{S} = \emptyset$, $f(\xi_t, \xi_T) = F^{-1}(1 - G(\xi_T))$.

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Existence of the optimum ⇔ Existence of minimum copula

Theorem (Sufficient condition for existence of a minimal copula L)

Let $\mathbb{S}$ be a **rectangle** $[u_1, u_2] \times [v_1, v_2] \subseteq [0,1]^2$. *Then a minimal copula $L(u,v)$ satisfying $\mathbb{S}$ exists and is given by*

$$L(u,v) = \max\{0,\ u+v-1,\ K(u,v)\}.$$

where $K(u,v) = \max_{(a,b)\in\mathbb{S}}\{Q(a,b) - (a-u)^+ - (b-v)^+\}$.

Proof in a note written with Xiao Jiang and Steven Vanduffel extending Tankov's result.

Consequently the existence of a $\mathbb{S}-$constrained cost-efficient payoff is guaranteed when $\mathbb{S}$ is a rectangle. More generally it also holds when $\mathbb{S} \subseteq [0,1]^2$ satisfies a "monotonicity property" of the upper and lower "boundaries" and

$$\forall (u, v_0), (u, v_1) \in \mathcal{S},\ \left(u, \frac{v_0+v_1}{2}\right) \in \mathcal{S}. \qquad (7)$$