# Optimal Portfolio Under Worst-Case Scenarios

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## Contributions

1. A better understanding of the link between Growth Optimal Portfolio and optimal investment strategies
2. Understanding issues with traditional diversification strategies and how **lowest outcomes of optimal strategies always happen in the worse states of the economy**.
3. Develop **innovative** strategies to cope with this observation.
4. Implications in terms of **assessing the risk and return** of a strategy and in terms of **reducing systemic risk**

**Part I:**

**Traditional**

**Diversification Strategies**

## Growth Optimal Portfolio (GOP)

- The **Growth Optimal Portfolio** (GOP) maximizes expected logarithmic utility from terminal wealth.
- It has the property that it **almost surely accumulates more wealth than any other strictly positive portfolios after a sufficiently long time**.
- Under general assumptions on the market, the GOP is a diversified portfolio.
- Details in Platen (2006).

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## For example, in the Black-Scholes model

- A Black-Scholes financial market (mainly for ease of exposition)
- Risk-free asset $\{B_t = B_0 e^{rt},\ t \geqslant 0\}$
- $$\begin{cases} \frac{dS_t^1}{S_t^1} = \mu_1 dt + \sigma_1 dW_t^1 \\ \frac{dS_t^2}{S_t^2} = \mu_2 dt + \sigma_2 dW_t \end{cases}, \tag{1}$$

  where $W^1$ and $W$ are two correlated Brownian motions under the physical probability measure $\mathbb{P}$.

  $$W_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$$

  where $W^1$ and $W^2$ are independent.

## Constant-Mix Strategy

- Dynamic rebalancing to preserve the initial target allocation
- The payoff of a constant-mix strategy is

$$S_t^{\pi} = S_0^{\pi} \ \exp(X_t^{\pi})$$

where $X_t^{\pi}$ is normal.

- For an initial investment $V_0$, $V_T$ is given by

$$V_T = V_0 \frac{S_T^{\pi}}{S_0^{\pi}},$$

where $\pi$ is the vector of proportions.

## Growth Optimal Portfolio (GOP)

In the 2-dimensional Black-Scholes setting,

- The GOP is a constant-mix strategy with $X_t^\pi = \left(\mu_\pi - \frac{1}{2}\sigma_\pi^2\right) t + \sigma_\pi W_t^\pi$, that **maximizes the expected growth rate** $\mu_\pi - \frac{1}{2}\sigma_\pi^2$. It is
$$\pi^\star = \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1})\,. \tag{2}$$
- **constant-mix portfolios** given by $\pi = \alpha\pi^\star$ with $\alpha > 0$ and where $\pi^\star$ is the optimal proportion for the GOP, are optimal strategies for CRRA expected utility maximizers. With a constant relative risk aversion coefficient $\eta > 0$, CRRA utility is
$$U(x) = \begin{cases} \frac{x^{1-\eta}}{1-\eta} & \text{when} \quad \eta \neq 1 \\ \log(x) & \text{when} \quad \eta = 1, \end{cases}$$
and $\alpha = 1/\eta$.

## Market Crisis

The **growth optimal portfolio** $S^\star$ can also be interpreted as a major market index. Hence it is intuitive to define a stressed market (or crisis) at time $T$ as an event where *the market* - materialized through $S^\star$ - **drops below its Value-at-Risk** at some high confidence level. The corresponding states of the economy verify

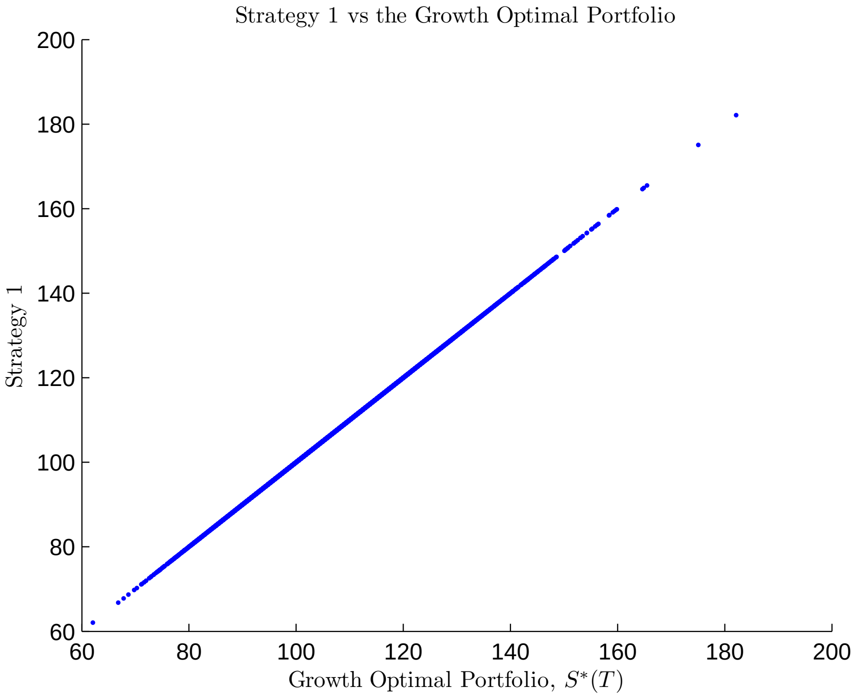
$$\textit{Crisis states} = \{S^\star_T < q_\alpha\}, \tag{3}$$

where $q_\alpha$ is such that $P(S^\star_T < q_\alpha) = 1 - \alpha$ and $\alpha$ is typically high (e.g. $\alpha = 0.98$).

## Srategy 1: GOP

We invest fully in the GOP.

In a crisis (GOP is low), our portfolio is low!


Strategy 1 vs the Growth Optimal Portfolio
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Growth Optimal Portfolio, $S^*(T)$

## Srategy 2: Buy-and-Hold

The buy-and-hold strategy is the simplest investment strategy. An initial amount $V_0$ is used to purchase $w_0$ units of the bank account and $w_i$ units of stock $S^i$ $(i = 1, 2)$ such that

$$V_0 = w_0 + w_1 \; S_0^1 + w_2 \; S_0^2,$$

and no further action is undertaken.

Example with 1/3 invested in each asset (bank, $S_1$ and $S_2$) on next slide.

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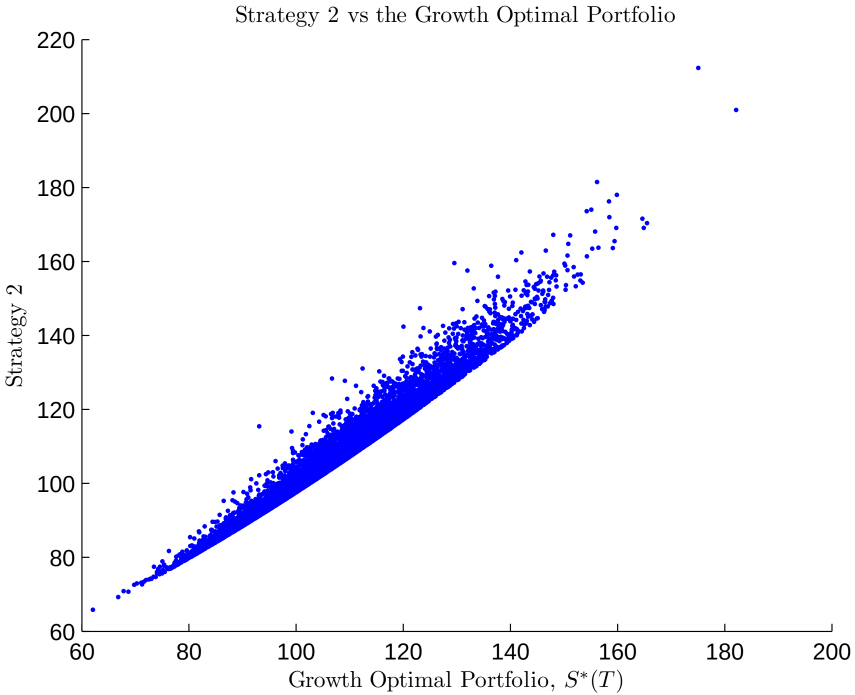
Strategy 2 vs the Growth Optimal Portfolio
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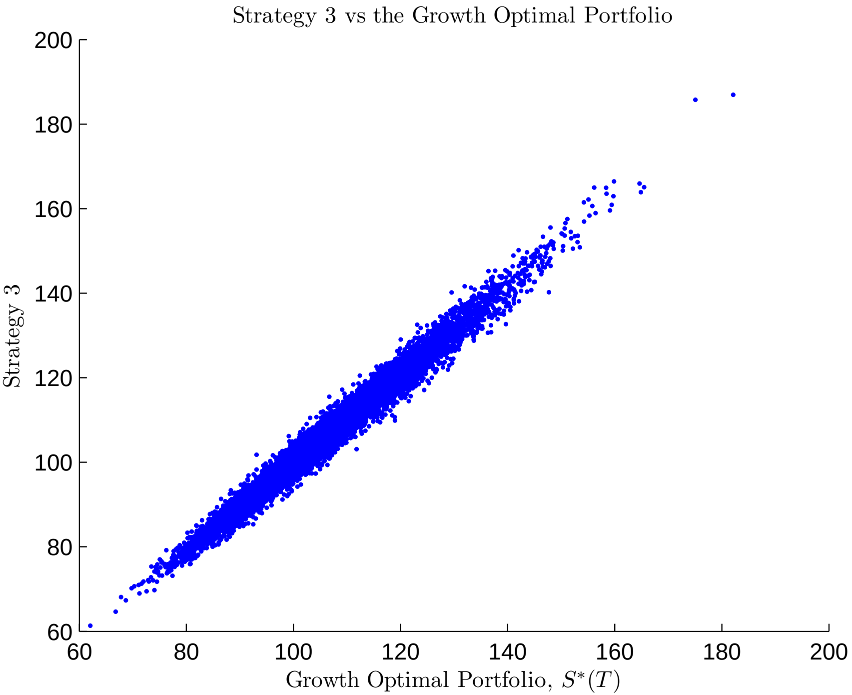
Growth Optimal Portfolio, $S^*(T)$

## Strategy 3: Constant-Mix Strategy

Example with 1/3 invested in each asset (bank, $S_1$ and $S_2$).



Strategy 3 vs the Growth Optimal Portfolio
Strategy 3
Growth Optimal Portfolio, $S^*(T)$
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- These three traditional diversification strategies do not offer protection during a crisis.
- In a more general setting, optimal strategies share the same problem...

**Part II:**

**Optimal portfolio selection**

**for law-invariant preferences**

**Stochastic Discount Factor and Real-World Pricing**:

The GOP can be used as numeraire to price under $P$

$$\left\{ \begin{array}{c} \textit{Price of} \\ X_T \textit{ at } 0 \end{array} \right\} = E_Q[e^{-rT} X_T] = E_P[\xi_T X_T] = E_P\left[\frac{X_T}{S_T^\star}\right]$$

where $S_0^\star = 1$.

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where $S^\star_0 = 1$.

## Cost-efficient strategies (Dybvig (1988))

**Optimal Portfolio Selection Problem:** Consider an investor with **fixed investment horizon**:

$$\max_{\mathbf{X_T}} \mathcal{U}(\mathbf{X_T})$$

subject to a given "cost of $X_T$" (equal to initial wealth)

- **Law-invariant** preferences $X_T \sim Y_T \Rightarrow \mathcal{U}(X_T) = \mathcal{U}(Y_T)$
- **Increasing** preferences

$$X_T \sim F, Y_T \sim G, \forall x, F(x) \leqslant G(x) \Rightarrow \mathcal{U}(X_T) \geqslant \mathcal{U}(Y_T)$$

**A strategy (or a payoff) is cost-efficient**

if any other strategy that generates the same distribution under $P$ costs at least as much.

The optimal strategy for $\mathcal{U}$ must be **cost-efficient**.

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## Optimal Portfolio and Cost-efficiency

Consider an investor with **increasing law-invariant** preferences and a **fixed** horizon. Denote by $X_T$ the investor's final wealth. The optimal strategy solves a cost-efficiency problem

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\frac{X_T}{S_T^\star}\right]$$

**Reciprocally** a cost-efficient strategy with a continuous distribution $F$ corresponds to the optimum of an expected utility investor for

$$U(x) = \int_0^x G^{-1}(1 - F(y))dy$$

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# Black-Scholes Model

## Theorem

*Consider the following optimization problem:*

$$PD(F) := \min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\frac{X_T}{S_T^\star}\right]$$

*In a Black-Scholes model, the optimal strategy (cheapest way to get $F$) is*

$$X_T^\star = F^{-1}\left(F_{S_T^\star}\left(S_T^\star\right)\right).$$

*Note that $X_T^\star \sim F$ and $X_T^\star$ is a.s.* **unique**.

## Corollary

*A strategy with payoff $X_T = h(S_T^\star)$ is cost-efficient if and only if $h$ is non-decreasing.*

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## Idea of the proof

$$\min_{X_T} \; \mathbb{E}\left[\frac{X_T}{S_T^\star}\right]$$
$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \frac{1}{S_T^\star} \sim G \end{cases}$$

Recall that

$$\text{corr}\left(X_T, \frac{1}{S_T^\star}\right) = \frac{\mathbb{E}\left[X_T \frac{1}{S_T^\star}\right] - \mathbb{E}[\frac{1}{S_T^\star}]\mathbb{E}[X_T]}{\text{std}(\frac{1}{S_T^\star})\,\text{std}(X_T)}.$$

We can prove that when the distributions for both $X_T$ and $\frac{1}{S_T^\star}$ are fixed, we have

$$\mathbf{(X_T, S_T^\star)} \text{ is comonotonic} \Rightarrow \text{corr}\left[\mathbf{X_T}, \frac{\mathbf{1}}{\mathbf{S_T^\star}}\right] \text{ is minimal.}$$

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# Part III:

# Investment under Worst-Case Scenarios

## Investment with State-Dependent Constraints

Problem considered so far

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\frac{X_T}{S_T^\star}\right].$$

A payoff that solves this problem is **cost-efficient**.

New Problem

$$\min_{\{V_T \mid V_T \sim F, \mathbb{S}\}} \mathbb{E}\left[\frac{V_T}{S_T^\star}\right].$$

where $\mathbb{S}$ denotes a set of constraints. A payoff that solves this problem is called a $\mathbb{S}-$**constrained cost-efficient payoff**.

## Type of Constraints

We are able to find optimal strategies with final payoff $V_T$

- with an additional probability constraint

$$P(S_T^\star \leqslant s, V_T \leqslant v) = \beta$$

- with a set of probability constraints

$$\forall (s,v) \in \mathbb{S},\ P(S_T^\star \leqslant s, V_T \leqslant v) = Q(s,v)$$

where $Q$ is an appropriate given function and $\mathbb{S}$ verifies some properties.

- in particular, assuming that the final payoff of the strategy is independent of $S_T^\star$ during a crisis (defined as $S_T^\star \leqslant q_\alpha$),

$$\forall s \leqslant q_\alpha, v \in \mathbb{R}, P(S_T^\star \leqslant s, V_T \leqslant v) = P(S_T^\star \leqslant s) P(V_T \leqslant v)$$

# Independence in the Tail - Strategy 4: Path-dependent

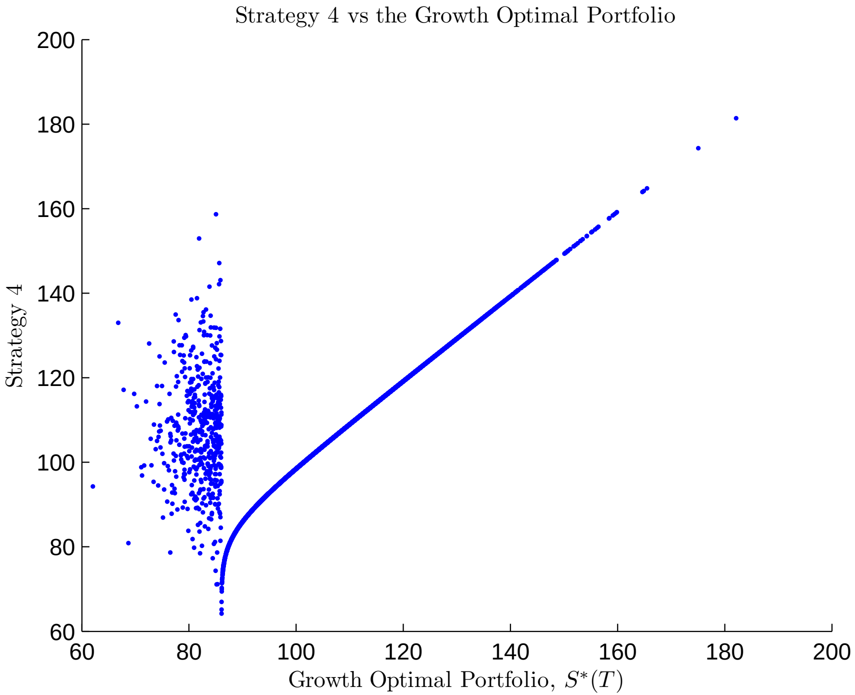
**Theorem**

*The cheapest path-dependent strategy with a cumulative distribution $F$ but such that it is independent of $S_T^\star$ when $S_T^\star \leqslant q_\alpha$ can be constructed as*

$$V_T^\star = \begin{cases} F^{-1}\left(\frac{F_{S_T^\star}(S_T^\star)-\alpha}{1-\alpha}\right) & \text{when} \quad S_T^\star > q_\alpha, \\ F^{-1}\left(\Phi\left(\frac{\ln\left(\frac{S_t^\star}{\left(S_T^\star\right)^{t/T}}\right)-(1-\frac{t}{T})\ln(S_0^\star)}{\sigma_\star\sqrt{t-\frac{t^2}{T}}}\right)\right) & \text{when} \quad S_T^\star \leqslant q_\alpha, \end{cases} \tag{4}$$

*where $t \in (0, T)$ can be chosen freely.*

(**No uniqueness and path-independence anymore**).


Strategy 4 vs the Growth Optimal Portfolio
Strategy 4
Growth Optimal Portfolio, $S^*(T)$
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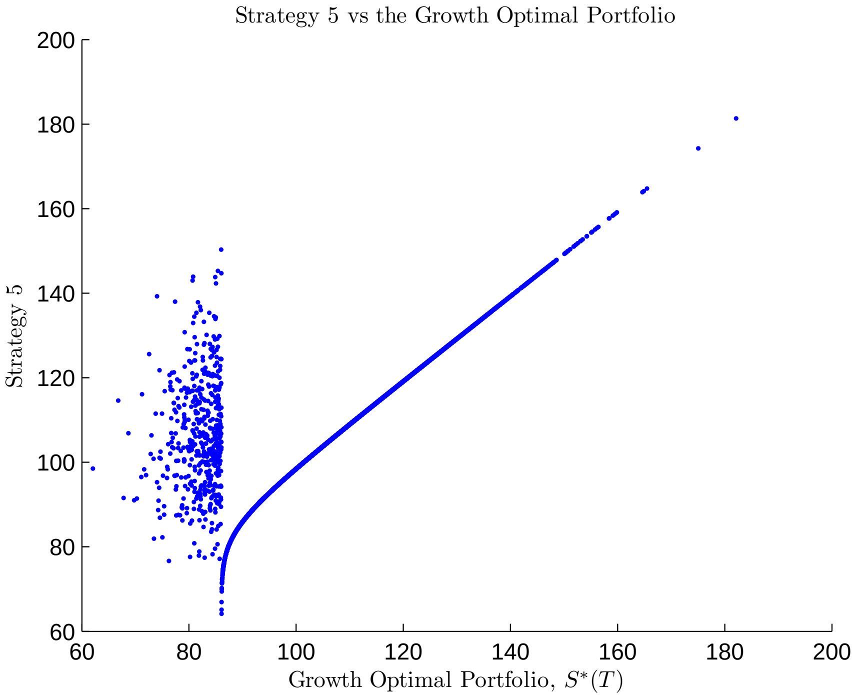
## Independence in the Tail - Strategy 5: Path-independent

In a financial market that **contains at least two assets** that are continuously distributed, the **cheapest** path-independent strategy with a cumulative distribution $F$ but such that it is **independent** of $S_T^\star$ when $S_T^\star \leqslant q_\alpha$ can be constructed as

$$Z_T^\star = \begin{cases} F^{-1}\left(\frac{F_{S_T^\star}(S_T^\star)-\alpha}{1-\alpha}\right) & \text{when} \quad S_T^\star > q_\alpha \\ F^{-1}(\Phi(A)) & \text{when} \quad S_T^\star \leqslant q_\alpha \end{cases} . \tag{5}$$

where $A$ is explicitly known as a function of $S_T^1$ and $S_T^\star$.


Strategy 5 vs the Growth Optimal Portfolio
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**Part IV:**

**Investment under**

**Worst-Case Scenarios**

**Some numerical examples**

## Other Types of Dependence

Recall that the joint cdf of a couple $(S_T^\star, X)$ writes as

$$P(S_T^\star \leqslant s, X_T \leqslant x) = C(H(s), F(x))$$

where

- The marginal cdf of $S_T^\star$: $H$
- The marginal cdf of $X_T$: $F$
- A copula $C$

**Independence** in the tail (independence copula $C(u,v) = uv$):

$$\forall s \leqslant q_\alpha, v \in \mathbb{R}, P(S_T^\star \leqslant s, V_T \leqslant v) = P(S_T^\star \leqslant s) P(V_T \leqslant v)$$

- We were also able to derive formulas for optimal strategies that generate a **Gaussian distribution** in the tail with a correlation coefficient of -0.5.
- Similarly for **Clayton** or **Frank** dependence.

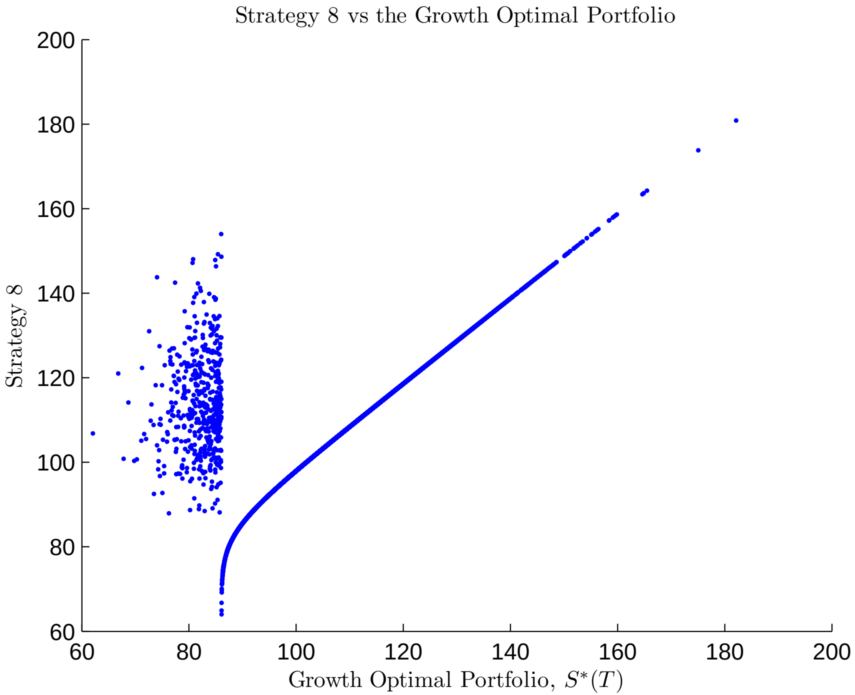
## Optimal Investment with a Clayton Tail Dependence

The **cheapest** strategy $V_T^\star$ with cdf $F$ that verifies this **Clayton** dependence (with correlation -0.5) in the tail is

$$V_T^\star = \begin{cases} F^{-1}\left(\left[(F_{S_T^\star}(S_T^\star)-\alpha)^{-a}-(1-\alpha)^{-a}+1\right]^{-1/a}\right) & \text{if} \quad S_T^\star > q_\alpha \\ F^{-1}\left(g\left(1-F_{S_T^\star}(S_T^\star), j_{F_{S_T^\star}(S_T^\star)}(F_{Z_T}(Z_T))\right)\right) & \text{if} \quad S_T^\star \leqslant q_\alpha, \end{cases}$$

where $Z_T$ is such that $(S_T^\star, Z_T)$ is continuously distributed (with copula $J$) and where $g$ is known explicitly:

$$g(u,v) = \left[u^{-a}\left(v^{-a/(1+a)}-1\right)+1\right]^{-1/a}.$$


Strategy 8 vs the Growth Optimal Portfolio
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## Some numerical results

We define two events related to *the market*, i.e. the market **crisis** $\mathbf{C} = \{\mathbf{S}^{\star}_{\mathbf{T}} < \mathbf{q}_{\alpha}\}$ and a **decrease** in the market $\mathbf{D} = \{\mathbf{S}^{\star}_{\mathbf{T}} < \mathbf{S}^{\star}_{\mathbf{0}}\mathbf{e}^{\mathbf{rT}}\}$. We further define two events for the portfolio value by $A = \left\{V_T < V_0 e^{rT}\right\}$ and $B = \left\{V_T < 75\% V_0 e^{rT}\right\}$

| | $T$ | Cost | Sharpe | $P(A\|\mathbf{C})$ | $P(A\|\mathbf{D})$ | $P(B\|\mathbf{C})$ |
|---|---|---|---|---|---|---|
| GOP | 5 | 100 | 0.266 | 1.00 | 1.00 | 1.00 |
| Buy-and-Hold | 5 | 100 | 0.239 | 0.9998 | 0.965 | 0.99 |
| Independence | 5 | 101.67 | 0.214 | 0.46 | 0.94 | 0.13 |
| Gaussian | 5 | 103.40 | 0.159 | 0.12 | 0.90 | 0.01 |
| Clayton | 5 | 102.35 | 0.193 | 0.24 | 0.91 | 0.02 |

## Conclusions

- **Cost-efficiency:** a preference-free framework for ranking different investment strategies.
- **Characterization of optimal portfolio strategies** for investors with law invariant preferences and a fixed horizon.

▶ **Lowest outcomes in worst states** of the economy

- Optimal investment choice under **state-dependent** constraints.
  - not always non-decreasing with the GOP $S_T^\star$.
  - not anymore unique
  - could be path-dependent.

▶ **Trade-off** between losing "utility" and gaining from better fit of the investor's preferences.

# More Implications

- The new strategies do not incur their biggest losses in the worst states in the economy.
- can be used to **reduce systemic risk**.
  - the idea of assessing risk and performance of a portfolio not only by looking at its final distribution but also by looking at its interaction with the economic conditions is indeed related to the increasing concern to evaluate systemic risk.
  - Acharya (2009) explains that regulators should "be regulating each bank as a function of both its joint (correlated) risk with other banks as well as its individual (bank-specific) risk".
  - An insight of this work is that if all institutional investors implement strategies that are resilient against crisis regimes, as we propose, then systemic risk can be diminished.

*Do not hesitate to contact me to get updated working papers!*

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# Part V:

# Proofs with Copulas

# Optimal Portfolio under Tail Dependence

## Copulas and Sklar's theorem

The joint cdf of a couple $(\xi_T, X)$ can be decomposed into 3 elements

- The marginal cdf of $\xi_T$: $G$
- The marginal cdf of $X_T$: $F$
- A copula $C$

such that

$$P(\xi_T \leqslant \xi, X_T \leqslant x) = C(G(\xi), F(x))$$

## Where do copulas appear?

in the derivation of "cost-efficient" strategies...

Solving the cost-efficiency problem amounts to finding bounds on copulas!

$$\min_{X_T} \mathbb{E}\left[\xi_T X_T\right]$$

$$\text{subject to} \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

## Proof of the cost-efficient payoff

$$\min_{X_T} \mathbb{E}[\xi_T X_T]$$

$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

The distribution $G$ is known and depends on the financial market. Let $C$ denote a copula for $(\xi_T, X)$.

$$\mathbb{E}[\xi_T X] = \int\int (1 - G(\xi) - F(x) + C(G(\xi), F(x)))dxd\xi, \quad (6)$$

The lower bound for $\mathbb{E}[\xi_T X]$ is derived from the lower bound on $C$

$$\max(u + v - 1, 0) \leqslant C(u, v)$$

(where $\max(u + v - 1, 0)$ corresponds to the **anti-monotonic** copula).

$$E[\xi_T F^{-1}(1 - G(\xi_T))] \leqslant E[\xi_T X_T]$$

**then $X_T^\star = F^{-1}(1 - G(\xi_T))$ has the minimum price for the cdf $F$.**

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$$\text{subject to} \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

The distribution $G$ is known and depends on the financial market. Let $C$ denote a copula for $(\xi_T, X)$.

$$\mathbb{E}[\xi_T X] = \int\int (1 - G(\xi) - F(x) + C(G(\xi), F(x)))dxd\xi, \quad (6)$$

The lower bound for $\mathbb{E}[\xi_T X]$ is derived from the lower bound on $C$

$$\max(u+v-1,0) \leqslant C(u,v)$$

(where $\max(u+v-1,0)$ corresponds to the **anti-monotonic** copula).

$$E[\xi_T F^{-1}(1-G(\xi_T))] \leqslant E[\xi_T X_T]$$

**then $X_T^\star = F^{-1}(1-G(\xi_T))$ has the minimum price for the cdf $F$.**

## Proof of the cost-efficient payoff

$$\min_{X_T} \mathbb{E}\left[\xi_T X_T\right]$$

$$\text{subject to } \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

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## Sufficient condition for the existence

**Theorem**

*Let $t \in (0, T)$. If there exists a copula $L$ satisfying $\mathbb{S}$ such that $L \leqslant C$ (pointwise) for all other copulas $C$ satisfying $\mathbb{S}$ then the payoff $Y_T^\star$ given by*

$$Y_T^\star = F^{-1}(f(\xi_T, \xi_t))$$

*is a $\mathbb{S}$-constrained cost-efficient payoff. Here $f(\xi_T, \xi_t)$ is given by*

$$f(\xi_T, \xi_t) = \left(\ell_{G(\xi_T)}\right)^{-1}\left[j_{G(\xi_T)}(G(\xi_t))\right],$$

*where the functions $j_u(v)$ and $\ell_u(v)$ are defined as the first partial derivative for $(u, v) \to J(u, v)$ and $(u, v) \to L(u, v)$ respectively and where $J$ denotes the copula for the random pair $(\xi_T, \xi_t)$.*

If $(U, V)$ has a copula $L$ then $\ell_u(v) = \mathbb{P}(V \leqslant v | U = u)$.

When $\mathbb{S} = \emptyset$, $f(\xi_t, \xi_T) = F^{-1}(1 - G(\xi_T))$.

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# Existence of the optimum ⇔ Existence of minimum copula

**Theorem (Sufficient condition for existence of a minimal copula $L$)**

*Let $\mathbb{S}$ be a* **rectangle** $[u_1, u_2] \times [v_1, v_2] \subseteq [0,1]^2$. *Then a minimal copula $L(u,v)$ satisfying $\mathbb{S}$ exists and is given by*

$$L(u,v) = \max\left\{0,\ u+v-1,\ K(u,v)\right\}.$$

*where* $K(u,v) = \max_{(a,b)\in\mathbb{S}}\left\{Q(a,b) - (a-u)^+ - (b-v)^+\right\}$.

Proof in a note written with Xiao Jiang and Steven Vanduffel extending Tankov's result.

**Consequently the existence of a $\mathbb{S}-$constrained cost-efficient payoff is guaranteed when $\mathbb{S}$ is a rectangle. More generally it also holds when $\mathbb{S} \subseteq [0,1]^2$ satisfies a "monotonicity property" of the upper and lower "boundaries" and**

$$\forall\,(u,v_0),(u,v_1)\in\mathcal{S},\ \left(u,\frac{v_0+v_1}{2}\right)\in\mathcal{S}. \tag{7}$$