Optimal Portfolio Under Worst-Case Scenarios

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Contributions

1. Understanding issues with traditional diversification strategies (buy-and-hold, constant mix, Growth Optimal Portfolio) and how **lowest outcomes of optimal strategies always happen in the worst states of the economy**.
2. Develop **innovative** strategies to cope with this observation.
3. Implications in terms of **assessing risk and return** of a strategy and in terms of **reducing systemic risk**

Part I:

Traditional

Diversification Strategies

Growth Optimal Portfolio (GOP)

- The **Growth Optimal Portfolio** (GOP) maximizes expected logarithmic utility from terminal wealth.
- It has the property that it **almost surely accumulates more wealth than any other strictly positive portfolios after a sufficiently long time**.
- Under general assumptions on the market, the GOP is a diversified portfolio.
- Details in Platen & Heath (2006).

For example, in the 2-dim Black-Scholes model

- Risk-free asset $\{B_t = B_0 e^{rt},\ t \geqslant 0\}$

$$\begin{cases} \frac{dS_t^1}{S_t^1} = \mu_1 dt + \sigma_1 dW_t^1 \\ \frac{dS_t^2}{S_t^2} = \mu_2 dt + \sigma_2 dW_t \end{cases},$$

  where W^1 and W are two correlated Brownian motions under the physical probability measure $\mathbb{P}$.
- **Constant-mix strategy:** Dynamic rebalancing to preserve the initial target allocation. The payoff of a constant-mix strategy is $S_t^\pi = S_0^\pi \ \exp(X_t^\pi)$ where X_t^π is normal.
- The **Growth Optimal Portfolio (GOP)** is a constant-mix strategy with $X_t^\pi = \left(\mu_\pi - \frac{1}{2}\sigma_\pi^2\right) t + \sigma_\pi W_t^\pi$, that **maximizes the expected growth rate** $\mu_\pi - \frac{1}{2}\sigma_\pi^2$. It is

$$\pi^\star = \mathbf{\Sigma}^{-1} \cdot \left(\overrightarrow{\mu} - r\ \overrightarrow{\mathbb{1}}\right).$$

Market Crisis

The **growth optimal portfolio** $S^\star$ can also be interpreted as a major market index. Hence it is intuitive to define a stressed market (or crisis) at time T as an event where *the market* - materialized through $S^\star$ - **drops below its Value-at-Risk** at some high confidence level. The corresponding states of the economy verify

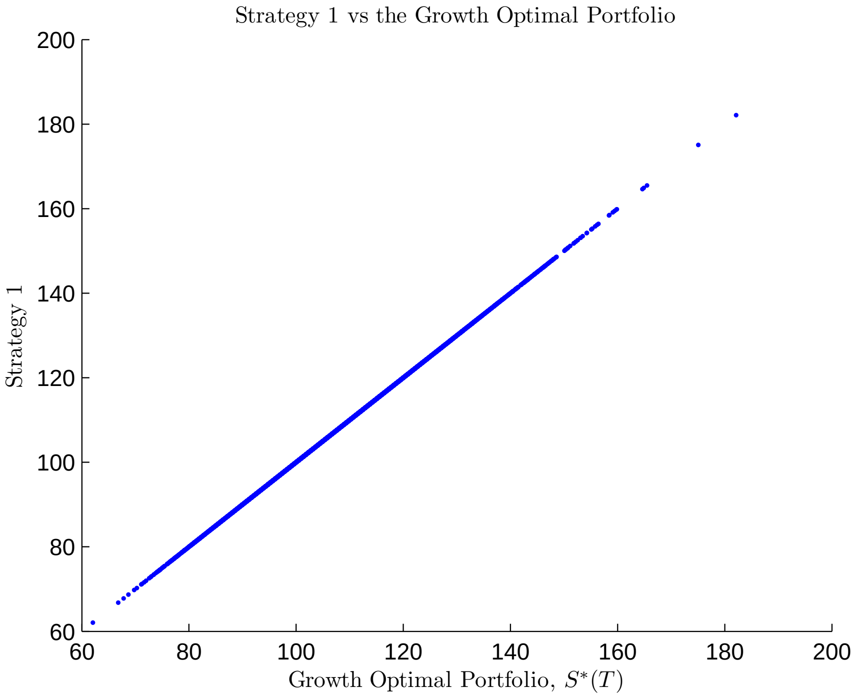
$$\textit{Crisis states} = \{S^\star_T < q_\alpha\}\,, \tag{1}$$

where q_α is such that $P(S^\star_T < q_\alpha) = 1 - \alpha$ and α is typically high (e.g. $\alpha = 0.98$).

Strategy 1: GOP

We invest fully in the GOP.

In a crisis (GOP is low), our portfolio is low!



Strategy 1 vs the Growth Optimal Portfolio
200
180
160
140
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Strategy 1
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200
Growth Optimal Portfolio, S*(T)

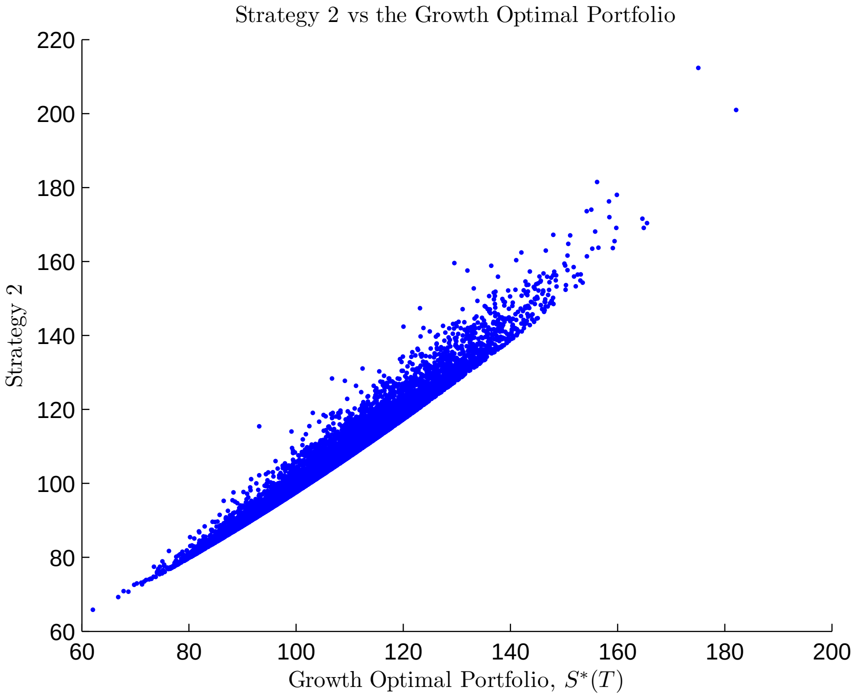
Strategy 2: Buy-and-Hold

The buy-and-hold strategy is the simplest investment strategy. An initial amount V_0 is used to purchase w_0 units of the bank account and w_i units of stock S^i $(i = 1, 2)$ such that

$$V_0 = w_0 + w_1\ S_0^1 + w_2\ S_0^2,$$

and no further action is undertaken.

Example with $1/3$ invested in each asset (bank, S_1 and S_2) on next slide.



Strategy 2 vs the Growth Optimal Portfolio
Strategy 2
Growth Optimal Portfolio, $S^*(T)$
60
80
100
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200
220

- These traditional diversification strategies do not offer protection during a crisis.
- In a more general setting, optimal strategies share the same problem...

Optimal Portfolio Selection Problem: Consider an investor with **fixed investment horizon**:

$$\max_{\mathbf{X_T}} \mathcal{U}(\mathbf{X_T})$$

subject to a given "cost of X_T" (equal to initial budget)

- **Law-invariant** preferences $X_T \sim Y_T \Rightarrow \mathcal{U}(X_T) = \mathcal{U}(Y_T)$
- **Increasing** preferences

$$X_T \sim F, Y_T \sim G, \forall x, F(x) \leqslant G(x) \Rightarrow \mathcal{U}(X_T) \geqslant \mathcal{U}(Y_T)$$

Optimal Investment

$$\max_{\{X_T \ / \ initial \ \ budget = x_0\}} \mathcal{U}(X_T)$$

Theorem

Optimal strategies for $\mathcal{U}$ must be **cost-efficient**.

where we recall the definition of cost-efficiency.

Definition - Dybvig (1988)

A strategy (or a payoff) is **cost-efficient** if any other strategy that generates the same distribution F under P costs at least as much.

A cost-efficient strategy solves the following optimization problem

$$\begin{array}{l} \min_{X_T} \ cost(X_T) \\ \text{subject to} \ \ \{X_T \sim F\} \end{array}.$$

Stochastic Discount Factor and Real-World Pricing:

The GOP can be used as numeraire to price under P

$$\left\{ \begin{array}{c} x_0 = Cost\ \ of \\ X_T\ \ at\ 0 \end{array} \right\} = E_Q[e^{-rT}X_T] = E_P\left[\frac{X_T}{S^\star_T}\right]$$

where $S^\star_0 = 1$.

Cost-efficiency problem: $\min_{X_T}\ E_P\left[\frac{X_T}{S^\star_T}\right]$ subject to $\{X_T \sim F\}$

Theorem

A strategy is cost-efficient **if and only if** its payoff is equal to $X_T = h(S^\star_T)$ where h is non-decreasing.

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A strategy is cost-efficient **if and only if** its payoff is equal to $X_T = h(S_T^\star)$ where h is non-decreasing.

Idea of the proof (First method)

$$\min_{X_T} \ \mathbb{E}\left[\frac{X_T}{S_T^\star}\right]$$
$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \frac{1}{S_T^\star} \sim G \end{cases}$$

Recall that

$$\text{corr}\left(X_T, \frac{1}{S_T^\star}\right) = \frac{\mathbb{E}\left[\frac{X_T}{S_T^\star}\right] - \mathbb{E}[\frac{1}{S_T^\star}]\mathbb{E}[X_T]}{\text{std}(\frac{1}{S_T^\star})\,\text{std}(X_T)}.$$

We can prove that when the distributions for both X_T and $\frac{1}{S_T^\star}$ are fixed, we have

$$\left(\mathbf{X_T}, \frac{\mathbf{1}}{\mathbf{S_T^\star}}\right) \text{ is anti-monotonic } \Leftrightarrow \text{corr}\left(\mathbf{X_T}, \frac{\mathbf{1}}{\mathbf{S_T^\star}}\right) \text{ is minimal.}$$

Idea of the proof (Second method)

Recall that the joint cdf of a couple $(S_T^\star, X_T)$ writes as

$$P(S_T^\star \leqslant s, X_T \leqslant x) = C(G(s), F(x))$$

where G is the marginal cdf of $S_T^\star$ (known: it depends on the financial market), F is the marginal cdf of X_T and C denotes the copula for $(S_T^\star, X)$.

$$\min_{X_T} \ \mathbb{E}\left[\frac{X_T}{S_T^\star}\right] \text{subject to } \ X_T \sim F$$

$$\mathbb{E}\left[\frac{X_T}{S_T^\star}\right] = \int\int (G(1/\xi) - C(G(1/\xi), F(x)))dxd\xi, \qquad (2)$$

The lower bound for $\mathbb{E}\left[\frac{X_T}{S_T^\star}\right]$ is derived from the upper bound on C

$$C(u,v) \leqslant \min(u,v)$$

(where $\min(u,v)$ corresponds to the **comonotonic** copula). **then** $X_T^\star = F^{-1}\left(G\left(S_T^\star\right)\right)$ **has the minimum price for the cdf** F**.**

Part II:

Investment under

Worst-Case Scenarios

Type of Constraints

We find optimal strategies with final payoff $X_T \sim F$

- with a set of probability constraints, for example assuming that the final payoff of the strategy is independent of $S_T^\star$ during a crisis (defined as $S_T^\star \leqslant q_\alpha$),

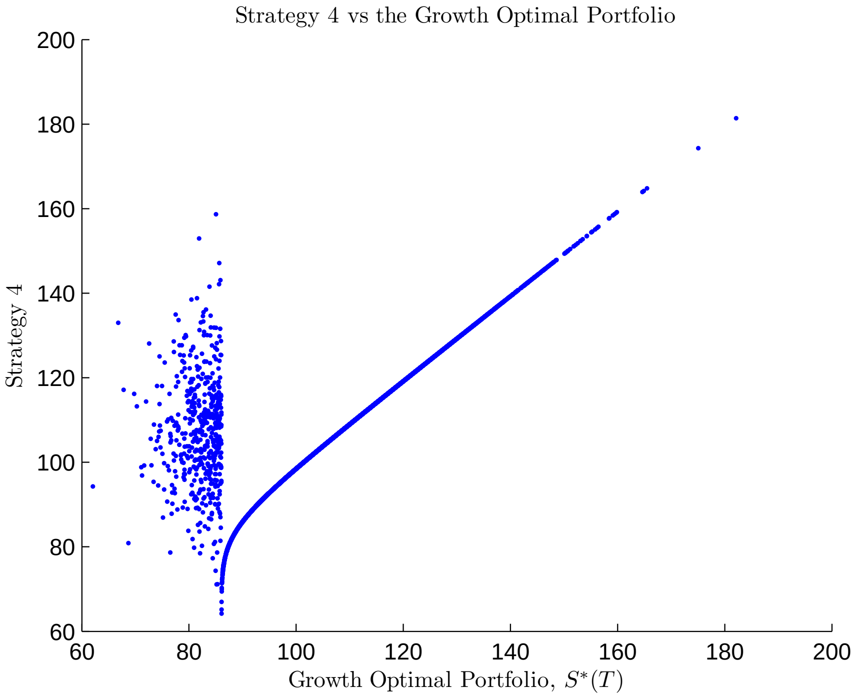
$$\forall s \leqslant q_\alpha, x \in \mathbb{R}, P(S_T^\star \leqslant s, X_T \leqslant x) = P(S_T^\star \leqslant s)P(X_T \leqslant x)$$

Theorem (Optimal Investment with Independence in the Tail)

The cheapest path-dependent strategy with cdf F and independent of $S_T^\star$ when $S_T^\star \leqslant q_\alpha$ can be constructed as

$$X_T^\star = \begin{cases} F^{-1}\left(\frac{F_{S_T^\star}(S_T^\star)-\alpha}{1-\alpha}\right) & \text{when} \quad S_T^\star > q_\alpha, \\ F^{-1}\left(g(S_t^\star, S_T^\star)\right) & \text{when} \quad S_T^\star \leqslant q_\alpha, \end{cases} \tag{3}$$

where $g(.,.)$ is explicit and $t \in (0, T)$ can be chosen freely.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Proof & Other Types of Dependence

Proof:

We make use of our results (JAP, 2012) extending Rachev and Rüschendorf (1998) and Tankov (JAP, 2011) to derive improved Fréchet bounds on copulas when there are constraints on a rectangle.

$$P(S_T^\star \leqslant s, X_T \leqslant x) = C(G(s), F(x))$$

where G is the marginal cdf of $S_T^\star$, F is the marginal cdf of X_T and C is a copula.

Optimal strategies can be derived explicitly:

- **Independence** in the tail ($C(u,v) = uv$):

$$\forall s \leqslant q_\alpha, x \in \mathbb{R}, P(S_T^\star \leqslant s, X_T \leqslant x) = P(S_T^\star \leqslant s)P(X_T \leqslant x)$$

- **Gaussian copula** in the tail with correlation -0.5.
- Similarly for **Clayton** or **Frank** dependence.

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Some numerical results

We define two events related to *the market*, i.e. the market **crisis** $\mathbf{C} = \{\mathbf{S}^\star_\mathbf{T} < \mathbf{q}_\alpha\}$. Define $\mathbf{A} = \left\{\mathbf{X_T} < \mathbf{x_0 e^{rT}}\right\}$.

	T	Cost	Sharpe	$P(\mathbf{A}\vert\mathbf{C})$
GOP	5	100	0.266	1.00
Buy-and-Hold	5	100	0.239	0.9998
Independence	5	101.67	0.214	0.46
Gaussian	5	103.40	0.159	0.12

Some Implications

- **Trade-off** between losing "utility" and gaining protection during a "crisis": the new strategies do not incur their biggest losses in the worst states in the economy.
- This can be used to **reduce systemic risk**.
  - the idea of assessing risk and performance of a portfolio not only by looking at its final distribution but also by looking at its interaction with the economic conditions is indeed related to the increasing concern to evaluate systemic risk.
  - Acharya (2009) explains that regulators should "be regulating each bank as a function of both its joint (correlated) risk with other banks as well as its individual (bank-specific) risk".
  - An insight of this work is that if all institutional investors implement strategies that are resilient against crisis regimes, as we propose, then systemic risk can be diminished.

Do not hesitate to contact me to get updated working papers!

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