Optimal Portfolio Under Worst-Case Scenarios

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in Insurance, Securities
& Quantitative Finance

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Contributions

1. A better understanding of the link between Growth Optimal Portfolio and optimal investment strategies
2. Understanding issues with traditional diversification strategies and how **lowest outcomes of optimal strategies always happen in the worse states of the economy**.
3. Develop **innovative** strategies to cope with this observation.
4. Implications in terms of **assessing the risk and return** of a strategy and in terms of **reducing systemic risk**

Part I:

Traditional

Diversification Strategies

Growth Optimal Portfolio (GOP)

- The **Growth Optimal Portfolio** (GOP) maximizes expected logarithmic utility from terminal wealth.
- It has the property that it **almost surely accumulates more wealth than any other strictly positive portfolios after a sufficiently long time**.
- Under general assumptions on the market, the GOP is a diversified portfolio.
- Details in Platen & Heath (2006).

For example, in the Black-Scholes model

- A Black-Scholes financial market (mainly for ease of exposition)
- Risk-free asset $\{B_t = B_0 e^{rt},\ t \geqslant 0\}$
- $$\begin{cases} \frac{dS_t^1}{S_t^1} = \mu_1 dt + \sigma_1 dW_t^1 \\ \frac{dS_t^2}{S_t^2} = \mu_2 dt + \sigma_2 dW_t \end{cases}, \tag{1}$$

  where W^1 and W are two correlated Brownian motions under the physical probability measure $\mathbb{P}$.

Constant-Mix Strategy

- Dynamic rebalancing to preserve the initial target allocation
- The payoff of a constant-mix strategy is

$$S_t^\pi = S_0^\pi \ \exp(X_t^\pi)$$

where X_t^π is normal.

- The **Growth Optimal Portfolio (GOP)** is a constant-mix strategy with $X_t^\pi = \left(\mu_\pi - \frac{1}{2}\sigma_\pi^2\right) t + \sigma_\pi W_t^\pi$, that **maximizes the expected growth rate** $\mu_\pi - \frac{1}{2}\sigma_\pi^2$. It is

$$\pi^\star = \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1}) \,. \qquad (2)$$

Market Crisis

The **growth optimal portfolio** $S^\star$ can also be interpreted as a major market index. Hence it is intuitive to define a stressed market (or crisis) at time T as an event where *the market* - materialized through $S^\star$ - **drops below its Value-at-Risk** at some high confidence level. The corresponding states of the economy verify

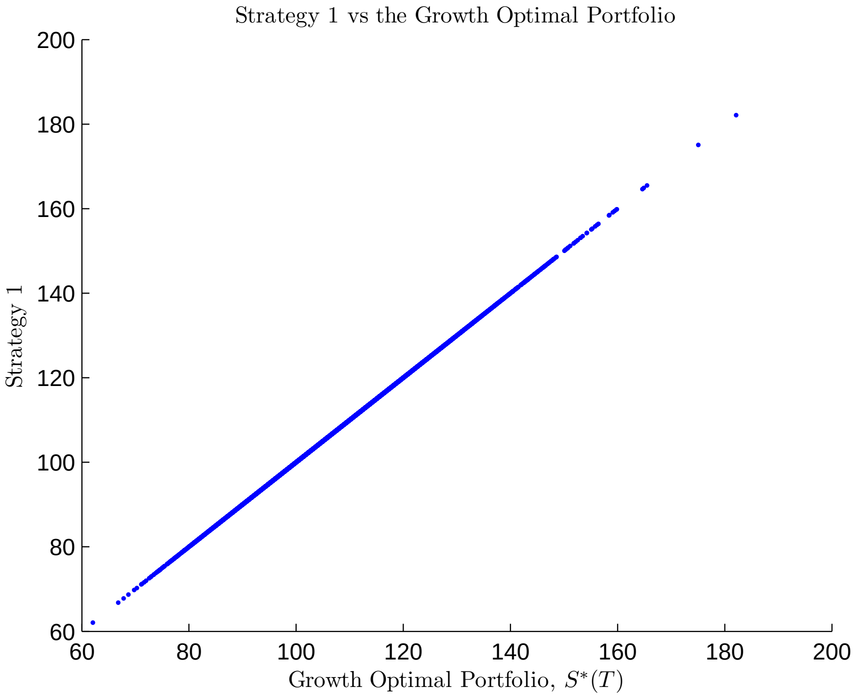
$$\textit{Crisis states} = \{S_T^\star < q_\alpha\}\,, \tag{3}$$

where q_α is such that $P(S_T^\star < q_\alpha) = 1 - \alpha$ and α is typically high (e.g. $\alpha = 0.98$).

Srategy 1: GOP

We invest fully in the GOP.

In a crisis (GOP is low), our portfolio is low!



Strategy 1 vs the Growth Optimal Portfolio
Strategy 1
Growth Optimal Portfolio, $S^*(T)$
200
180
160
140
120
100
80
60
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80
100
120
140
160
180
200

Srategy 2: Buy-and-Hold

The buy-and-hold strategy is the simplest investment strategy. An initial amount V_0 is used to purchase w_0 units of the bank account and w_i units of stock S^i $(i = 1, 2)$ such that

$$V_0 = w_0 + w_1 \ S_0^1 + w_2 \ S_0^2,$$

and no further action is undertaken.

Example with 1/3 invested in each asset (bank, S_1 and S_2) on next slide.

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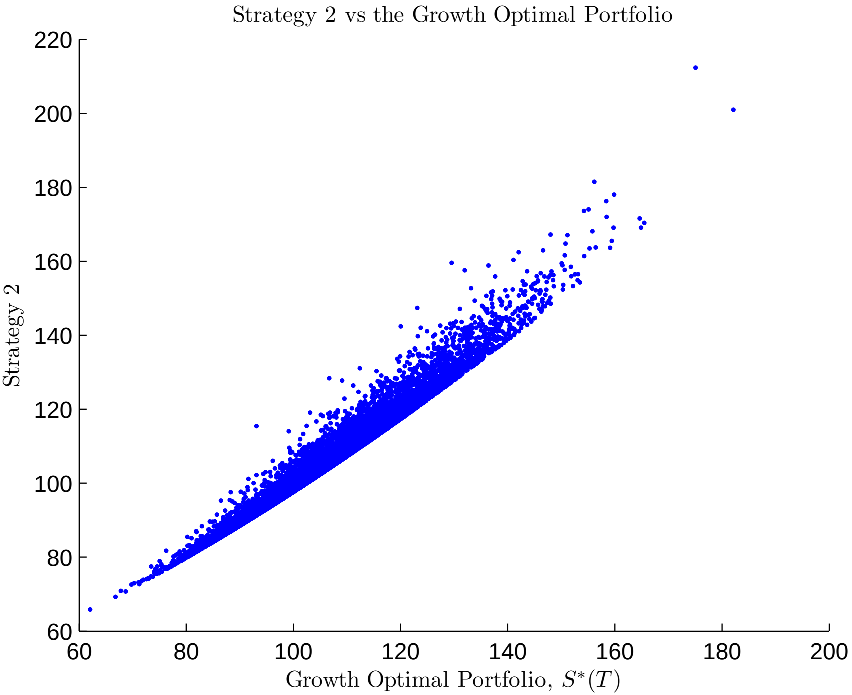
Strategy 2 vs the Growth Optimal Portfolio
220
200
180
160
140
120
100
80
60
Strategy 2
60
80
100
120
140
160
180
200
Growth Optimal Portfolio, $S^*(T)$

- These traditional diversification strategies do not offer protection during a crisis.
- In a more general setting, optimal strategies share the same problem...

Optimal Portfolio Selection Problem: Consider an investor with **fixed investment horizon**:

$$\max_{\mathbf{X_T}} \mathcal{U}(\mathbf{X_T})$$

subject to a given "cost of X_T" (equal to initial wealth)

- **Law-invariant** preferences $X_T \sim Y_T \Rightarrow \mathcal{U}(X_T) = \mathcal{U}(Y_T)$
- **Increasing** preferences

$$X_T \sim F, Y_T \sim G, \forall x, F(x) \leqslant G(x) \Rightarrow \mathcal{U}(X_T) \geqslant \mathcal{U}(Y_T)$$

Optimal Investment

Theorem

The optimal strategy for $\mathcal{U}$ must be **cost-efficient**.

Definition

A strategy (or a payoff) is **cost-efficient** if any other strategy that generates the same distribution under P costs at least as much.

Theorem

A strategy is cost-efficient **if and only if** *its payoff is equal to $X_T = h(S_T^\star)$ where h is non-decreasing.*

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Part II:

Investment under

Worst-Case Scenarios

Type of Constraints

We are able to find optimal strategies with final payoff V_T

- with a set of probability constraints, for example assuming that the final payoff of the strategy is independent of $S_T^\star$ during a crisis (defined as $S_T^\star \leqslant q_\alpha$),

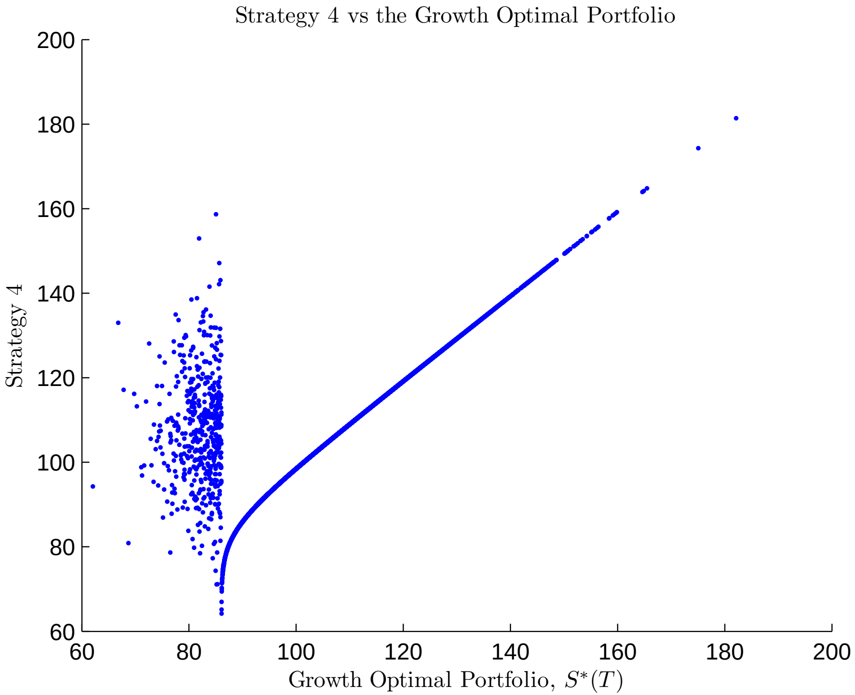
$$\forall s \leqslant q_\alpha, v \in \mathbb{R}, P(S_T^\star \leqslant s, V_T \leqslant v) = P(S_T^\star \leqslant s)P(V_T \leqslant v)$$

Theorem (Optimal Investment with Independence in the Tail)

The cheapest path-dependent strategy with cdf F and independent of $S_T^\star$ when $S_T^\star \leqslant q_\alpha$ can be constructed as

$$V_T^\star = \begin{cases} F^{-1}\left(\frac{F_{S_T^\star}(S_T^\star)-\alpha}{1-\alpha}\right) & when \quad S_T^\star > q_\alpha, \\ F^{-1}\left(g(S_t^\star, S_T^\star)\right) & when \quad S_T^\star \leqslant q_\alpha, \end{cases} \tag{4}$$

where $g(.,.)$ is explicit and $t \in (0, T)$ can be chosen freely.


Strategy 4 vs the Growth Optimal Portfolio
Strategy 4
Growth Optimal Portfolio, $S^*(T)$
60
80
100
120
140
160
180
200

Other Types of Dependence

Recall that the joint cdf of a couple $(S_T^\star, V_T)$ writes as

$$P(S_T^\star \leqslant s, V_T \leqslant x) = C(H(s), F(x))$$

where

- The marginal cdf of $S_T^\star$: H
- The marginal cdf of V_T: F
- A copula C

Independence in the tail (independence copula $C(u,v) = uv$):

$$\forall s \leqslant q_\alpha, v \in \mathbb{R}, P(S_T^\star \leqslant s, V_T \leqslant v) = P(S_T^\star \leqslant s)P(V_T \leqslant v)$$

- We were also able to derive formulas for optimal strategies that generate a **Gaussian copula** in the tail with a correlation coefficient of -0.5.
- Similarly for **Clayton** or **Frank** dependence.

Some numerical results

We define two events related to *the market*, i.e. the market **crisis** $\mathbf{C} = \{\mathbf{S}^\star_\mathbf{T} < \mathbf{q}_\alpha\}$ and a **decrease** in the market $\mathbf{D} = \{\mathbf{S}^\star_\mathbf{T} < \mathbf{S}^\star_\mathbf{0}\mathbf{e}^{\mathbf{rT}}\}$. We further define two events for the portfolio value by $A = \{V_T < V_0 e^{rT}\}$ and $B = \{V_T < 75\% V_0 e^{rT}\}$

	T	Cost	Sharpe	$P(A\|\mathbf{C})$	$P(A\|\mathbf{D})$	$P(B\|\mathbf{C})$
GOP	5	100	0.266	1.00	1.00	1.00
Buy-and-Hold	5	100	0.239	0.9998	0.965	0.99
Independence	5	101.67	0.214	0.46	0.94	0.13
Gaussian	5	103.40	0.159	0.12	0.90	0.01

Conclusions

- **Cost-efficiency:** a preference-free framework for ranking different investment strategies.
- **Characterization of optimal portfolio strategies** for investors with law invariant preferences and a fixed horizon.

▶ **Lowest outcomes in worst states** of the economy

- Optimal investment choice under **state-dependent** constraints.
  - not always non-decreasing with the GOP $S_T^\star$.
  - not anymore unique
  - could be path-dependent.

▶ **Trade-off** between losing "utility" and gaining from better fit of the investor's preferences.

More Implications

- The new strategies do not incur their biggest losses in the worst states in the economy.
- can be used to **reduce systemic risk**.
  - the idea of assessing risk and performance of a portfolio not only by looking at its final distribution but also by looking at its interaction with the economic conditions is indeed related to the increasing concern to evaluate systemic risk.
  - Acharya (2009) explains that regulators should “be regulating each bank as a function of both its joint (correlated) risk with other banks as well as its individual (bank-specific) risk”.
  - An insight of this work is that if all institutional investors implement strategies that are resilient against crisis regimes, as we propose, then systemic risk can be diminished.

Do not hesitate to contact me to get updated working papers!

References

- Bernard, C., Boyle P., Vanduffel S., 2011, "Explicit Representation of Cost-efficient Strategies", available on SSRN.
- Bernard, C., Jiang, X., Vanduffel, S., 2012. "Note on Improved Frechet bounds and model-free pricing of multi-asset options", *Journal of Applied Probability*.
- Bernard, C., Maj, M., Vanduffel, S., 2011. "Improving the Design of Financial Products in a Multidimensional Black-Scholes Market,", *North American Actuarial Journal*.
- Bernard, C., Vanduffel, S., 2011. "Optimal Investment under Probability Constraints," *AfMath Proceedings*.
- Bernard, C., Vanduffel, S., 2012. "Financial Bounds for Insurance Prices," *Journal of Risk Insurance*.
- Cox, J.C., Leland, H., 1982. "On Dynamic Investment Strategies," *Proceedings of the seminar on the Analysis of Security Prices*,(published in 2000 in *JEDC*).
- Dybvig, P., 1988a. "Distributional Analysis of Portfolio Choice," *Journal of Business*.
- Dybvig, P., 1988b. "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market," *Review of Financial Studies*.
- Goldstein, D.G., Johnson, E.J., Sharpe, W.F., 2008. "Choosing Outcomes versus Choosing Products: Consumer-focused Retirement Investment Advice," *Journal of Consumer Research*.
- Jin, H., Zhou, X.Y., 2008. "Behavioral Portfolio Selection in Continuous Time," *Mathematical Finance*.
- Nelsen, R., 2006. *"An Introduction to Copulas"*, Second edition, Springer.
- Pelsser, A., Vorst, T., 1996. "Transaction Costs and Efficiency of Portfolio Strategies," *European Journal of Operational Research*.
- Platen, E., 2005. "A benchmark approach to quantitative finance," Springer finance.
- Tankov, P., 2011. "Improved Frechet bounds and model-free pricing of multi-asset options," *Journal of Applied Probability, forthcoming*.
- Vanduffel, S., Chernih, A., Maj, M., Schoutens, W. 2009. "On the Suboptimality of Path-dependent Pay-offs in Lévy markets", *Applied Mathematical Finance*.

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