All Investors are Risk-averse Expected Utility Maximizers

Carole Bernard (UW), Jit Seng Chen (GGY) and Steven Vanduffel (Vrije Universiteit Brussel)

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Contributions

1. In any behavioral setting respecting **First-order Stochastic Dominance**, investors only care about the distribution of final wealth (**law-invariant** preferences).
2. In any such setting, the optimal portfolio is also the optimum for a **risk-averse Expected Utility maximizer**.
3. Given a distribution F of terminal wealth, we **construct a utility function** such that the optimal solution to

$$\max_{X_T \mid \text{budget}=\omega_0} E[U(X_T)]$$

has the cdf F.
4. Use this utility to **infer risk aversion**.
5. **Decreasing Absolute Risk Aversion** (DARA) can be directly related to properties of the distribution of final wealth and of the financial market in which the agent invests.

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FSD implies Law-invariance

Consider an investor with **fixed horizon** and objective $V(\cdot)$.

Theorem

Preferences $V(\cdot)$ are non-decreasing and law-invariant if and only if $V(\cdot)$ satisfies first-order stochastic dominance.

- **Law-invariant** preferences

$$X_T \sim Y_T \Rightarrow V(X_T) = V(Y_T)$$

- **Increasing** preferences

$$X_T \geqslant Y_T a.s. \Rightarrow V(X_T) \geqslant V(Y_T)$$

- **first-order stochastic dominance (FSD)**

$$X_T \sim F_X, Y_T \sim F_Y, \forall x, F_X(x) \leqslant F_Y(x) \Rightarrow V(X_T) \geqslant V(Y_T)$$

Main Assumptions

- Given a portfolio with final payoff X_T **(consumption only at time T)**.
- P ("physical measure"). The initial value of X_T is given by

$$\mathbf{c(X_T)} = \mathbb{E}_{\mathbf{P}}[\xi_{\mathbf{T}} \mathbf{X_T}].$$

  where ξ_T is called the pricing kernel.
- All market participants **agree** on ξ_T and ξ_T is **continuously distributed**.
- **Preferences satisfy FSD**.
- *Another approach:* Let Q be a **"risk-neutral measure"**, then

$$\xi_T = e^{-rT} \left(\frac{dQ}{dP} \right)_T, \quad c(X_T) = \mathbb{E}_Q[e^{-rT} X_T].$$

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Optimal Portfolio and Cost-efficiency

Definition:(Dybvig (1988), Bernard et al. (2011))

A payoff is cost-efficient if any other payoff that generates the same distribution under P costs at least as much.

Let X_T with cdf F. X_T is cost-efficient if it solves

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T] \tag{1}$$

The unique optimal solution to (1) is $X_T^\star = F^{-1}\left(1 - F_{\xi_T}\left(\xi_T\right)\right)$. Consider an investor with preferences **respecting FSD** and final wealth X_T at a **fixed** horizon.

Theorem 1:

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Optimal Portfolio and Cost-efficiency

Theorem 2:

An optimal payoff X_T with a continuous increasing distribution F also corresponds to the optimum of an expected utility investor for

$$U(x) = \int_0^x F_{\xi_T}^{-1}(1 - F(y))dy$$

where F_{ξ_T} is the cdf of ξ_T and budget$= E[\xi_T F^{-1}(1 - F_{\xi_T}(\xi_T))]$. The utility function U is C^1, strictly concave and increasing.

- When the optimal portfolio in a behavioral setting respecting FSD is continuously distributed, then it can be obtained by **maximum expected (concave) utility.**
- All distributions can be approximated by continuous distributions. Therefore all investors appear to be **approximately risk averse**...

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Generalization

We can show that all distributions can be the optimum of an expected utility optimization with a "generalized concave utility".

Definition: Generalized concave utility function

A generalized concave utility function $\widetilde{U} : \mathbb{R} \to \overline{\mathbb{R}}$ is defined as

$$\widetilde{U}(x) := \begin{cases} U(x) & \text{for } x \in (a,b), \\ -\infty & \text{for } x < a, \\ U(a^+) & \text{for } x = a, \\ U(b^-) & \text{for } x \geqslant b, \end{cases}$$

where $U(x)$ is concave and strictly increasing and $(a,b) \subset \mathbb{R}$.

General Distribution

Let F be

- a **continuous** distribution on (a, b)
- a **discrete** distribution on (m, M)
- a **mixed** distribution with $F = pF^d + (1-p)F^c$, $0 < p < 1$ and F^d (resp. F^c) is a discrete (resp. continuous) distribution.

Let $X_T^\star$ be the cost-efficient payoff for this cdf F. Assume its cost, ω_0, is finite. Then $X_T^\star$ is also an optimal solution to the following expected utility maximization problem

$$\max_{X_T \mid E[\xi_T X_T] = \omega_0} E\left[\widetilde{U}(X_T)\right]$$

where $\widetilde{U} : \mathbb{R} \to \overline{\mathbb{R}}$ is a generalized utility function given explicitly in the paper.

Illustration in the Black-Scholes model.

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P, \quad \frac{dB_t}{B_t} = rdt$$

Then

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T = a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = e^{\frac{\theta}{\sigma}(\mu - \frac{\sigma^2}{2})t - (r + \frac{\theta^2}{2})t}$, $\theta = \frac{\mu - r}{\sigma}$ and $b = \frac{\mu - r}{\sigma^2}$.

Utility & Distribution

- **Power utility (CRRA) & the LogNormal distribution:** $\mathcal{LN}(A, B^2)$ corresponds to a CRRA utility function with relative risk aversion $\frac{\theta\sqrt{T}}{B}$:

$$U(x) = \begin{cases} a\,\frac{x^{1-\frac{\theta\sqrt{T}}{B}}}{1-\frac{\theta\sqrt{T}}{B}} & \frac{\theta\sqrt{T}}{B} \neq 1, \\ a\,\log(x) & \frac{\theta\sqrt{T}}{B} = 1, \end{cases} \tag{2}$$

  where $a = \exp(\frac{A\theta\sqrt{T}}{B} - rT - \frac{\theta^2 T}{2})$.

- **Exponential utility & the Normal Distribution** $N(,)$ corresponds to the exponential utility $U(x) = -\exp(-\gamma x)$, where γ is the constant absolute risk aversion parameter.

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Explaining the Demand for Capital Guarantee Products

$$Y_T = \max(G, S_T)$$

where S_T is the stock price and G the guarantee.
$S_T \sim \mathcal{LN}(M_T, \Sigma_T^2)$.
The utility function is then given by

$$\widetilde{U}(x) = \begin{cases} -\infty & x < G, \\ a \, \frac{x^{1-\frac{\theta\sqrt{T}}{\Sigma_T}} - G^{1-\frac{\theta\sqrt{T}}{\Sigma_T}}}{1-\frac{\theta\sqrt{T}}{\Sigma_T}} & x \geqslant G, \ \frac{\theta\sqrt{T}}{\Sigma_T} \neq 1, \\ a \, \log(\frac{x}{G}) & x \geqslant G, \ \frac{\theta\sqrt{T}}{\Sigma_T} = 1, \end{cases} \tag{3}$$

with $a = \exp(\frac{M_T\theta\sqrt{T}}{\Sigma_T} - rT - \frac{\theta^2 T}{2})$.

- The mass point is explained by a utility which is infinitely negative for any level of wealth below the guaranteed level.
- The CRRA utility above this guaranteed level ensures the optimality of a Lognormal distribution above the guarantee.

Yaari's Dual Theory of Choice Model

Final wealth X_T. Objective function to maximize

$$\mathbb{H}_w[X_T] = \int_0^\infty w(1 - F(x))\,\mathrm{d}x,$$

where the (distortion) function $w : [0,1] \to [0,1]$ is non-decreasing with $w(0) = 0$ and $w(1) = 1$. Then, the optimal payoff is

$$X_T^\star = b\mathbb{1}_{\xi_T \leqslant c}$$

where $b > 0$ is given to fulfill the budget constraint.
We find that the utility function is given by

$$U(x) = \begin{cases} -\infty & x < 0 \\ f(x-c) & 0 \leqslant x \leqslant b \\ f(b-c) & x > b \end{cases}$$

where $f > 0$ is constant.

Inferring preferences and utility

- more natural for an investor to describe her target distribution than her utility (Goldstein, Johnson and Sharpe (2008) discuss how to estimate the distribution at retirement using a questionnaire).
- From the investment choice, get the distribution and find the corresponding utility U. $\Rightarrow$ Inferring preferences from the target final distribution
- $\Rightarrow$ Inferring risk-aversion. The Arrow-Pratt measure for absolute risk aversion can be computed from a twice differentiable utility function U as $\mathcal{A}(x) = -\frac{U''(x)}{U'(x)}$.
- Always possible to approximate by a twice differentiable utility function...

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Risk Aversion Coefficient

Theorem (Arrow-Pratt Coefficient)

Consider an investor who wants a cdf F (with density f). The Arrow-Pratt coefficient for absolute risk aversion is for $x = F^{-1}(p)$,

$$A(x) = \frac{f(F^{-1}(p))}{g(G^{-1}(p))},$$

where g and G are resp. the density and cdf of $-\log(\xi_T)$.

Theorem (Distributional characterization of DARA)

DARA iff $x \mapsto F^{-1}(G(x))$ is strictly convex.

In the special case of Black-Scholes: $x \mapsto F^{-1}(\Phi(x))$ is strictly convex, where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and cdf of N(0,1).

Some comments

- In the Black Scholes setting, DARA if and only if the target distribution F is fatter than a normal one.
- In a general market setting, DARA if and only if the target distribution F is fatter than the cdf of $-\log(\xi_T)$.
- Sufficient property for DARA:
  - logconvexity of $1-F$
  - decreasing hazard function ($h(x) := \frac{f(x)}{1-F(x)}$)
- Many cdf seem to be DARA even when they do not have decreasing hazard rate function. ex: Gamma, LogNormal, Gumbel distribution.

Conclusions & Future Work

- **Inferring** preferences and risk-aversion from the choice of distribution of terminal wealth.
- **Understanding** the interaction between changes in the financial market, wealth level and utility on optimal terminal consumption for an agent with given preferences.
- **FSD** or law-invariant behavioral settings **cannot explain** all decisions. One needs to look at state-dependent preferences to explain investment decisions such as
  - Buying protection...
  - Investing in highly path-dependent derivatives...

Do not hesitate to contact me to get updated working papers!

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