Dynamic Preferences for Popular Investment Strategies in Pension Funds

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Outline

1. Motivation & Contributions
2. Dynamic preferences: “Forward utility”
3. Dynamic Preferences for CPPI
4. Dynamic Preferences for Life-cycle Funds
5. Conclusions

Motivation

Utility function

- The way we measure satisfaction from consumption or wealth
- *Increasing* function : economic agent prefers a higher level of consumption or wealth to lower one.
- *Concave* function : marginal utility is decreasing

Classical optimal portfolio choice problem

Choose a utility function $\Rightarrow$ Find the optimal investment strategy

Opposite way

Given an investment strategy $\Rightarrow$ Infer the utility for it to be optimal?

Contributions

- Infer the utility for a dynamic strategy:
  - no specific horizon
  - the type of strategy is associated to a class of utility.
  - the parameters of the strategy are related to the risk aversion level.
- Work specifically on 2 examples **CPPI strategies** and **Life Cycle Funds**
- A standard **CPPI** strategy is optimal in a Black-Scholes model for HARA utility but it needs to have a dynamically updated multiple to be optimal for a HARA utility in a more general market.
- Some type of **life-cycle funds** can be optimal for the SAHARA utility (optimality of a decreasing proportion in risky asset over time). However, a constant decrease over time may not be optimal.

Strategy ⇒ Utility : Literature Review

Similar perspective, but different approach

- Dybvig and Rogers (1997) : “Recovery of Preferences from Observed Wealth in a Single Realization”
- Cuoco and Zapatero (2000) : “On the Recoverability of Preferences and Beliefs”
- Cox, Hobson, and Obloj. (2012) : “Utility Theory Front to Back - Inferring Utility from Agents’ Choices”
- Bernard, Chen, Vanduffel (2013): “All Investors are Risk Averse Expected Utility Maximizers”

Forward investment performance or Forward utility

- Musiela and Zariphopoulou (2009, 2010, 2011)
- Berrier, Rogers, and Tehranchi. (2010)

Outline

Forward Utility

1. Define “Forward Utility”
2. Illustrate Key Idea to find the forward utility

CPPI Strategy

1. Introduce CPPI strategy
2. Find the corresponding “Forward Utility” (which is a HARA utility at fixed time) corresponds to CPPI strategy

Life-Cycle Funds

1. Introduce Life-Cycle Funds
2. Introduce SAHARA utility
3. Find the corresponding “Forward Utility” (which is a SAHARA utility at fixed time) and corresponding investment strategy which is a kind of Life-Cycle Funds

Financial Market & Portfolio Value Process

- One-dimensional market with two assets: a risky asset S_t and a risk-free bond B_t

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t),\ S_0 > 0, \quad dB_t = r_t B_t dt,\ B_0 = 1,$$

- r_t, μ_t and σ_t may be stochastic but are adapted to the filtration $\mathcal{F}_t$
- Market price of risk (or instantaneous Sharpe ratio)

$$\lambda_t \triangleq \frac{\mu_t - r_t}{\sigma_t}$$

- Risk-free bond B_t is used as numéraire. Then, X_t^π : present value(value at time 0) of the portfolio at time t, with strategy π

$$X_t^\pi = \pi_t^0 + \pi_t$$

  - π_t^0 amount invested in the risk-free asset B_t
  - π_t amount invested in the risky asset S_t.
- Since B_t is used as numéraire,

$$d\pi_t^0 = 0, \quad dX_t^\pi = d\pi_t = \pi_t[(\mu_t - r_t)dt + \sigma_t dW_t] = \sigma_t \pi_t(\lambda_t dt + dW_t).$$

Definition of Forward Utility

Definition 2.1 (Forward utility)

An $\mathcal{F}_t$-adapted process $U_t(x)$ is a "Forward utility" if :

1. $x \to U_t(x)$ *is strictly concave and increasing*
2. *for each* $\pi \in \mathcal{A}$ *(i.e. for each attainable* X_s^π*), and* $t \geq s$,

$$\mathbb{E}[U_t(X_t^\pi)|\mathcal{F}_s] \leq U_s(X_s^\pi),$$

3. *there exists* $\pi^* \in \mathcal{A}$, *for which for all* $t \geq s$,

$$\mathbb{E}[U_t(X_t^{\pi^*})|\mathcal{F}_s] = U_s(X_s^{\pi^*}),$$

for $t \geq 0$ *and* $x \in \mathbb{D}$ *where* $\mathbb{D}$ *is an interval of* $\mathbb{R}$

Explanation for the Definition of Forward Utility

- For a fixed t, $x \to U_t(x)$ is a concave, increasing function.
- For some $T > 0$, let us define $v(x,t)$ as

$$v(x,t) \triangleq \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U_T(X_T^\pi) | \mathcal{F}_t, X_t^\pi = x\right] \tag{1}$$

  where $U_t(x)$ is a forward utility defined in the previous page.
- Let $\pi \in \mathcal{A}$ and π^* is the optimum. Then, by dynamic programming principle,

$$(v(X_s^\pi, s))_s : \text{ Supermartingale for each } \pi$$

$$(v(X_s^{\pi^*}, s))_s : \text{ Martingale for } \pi^*$$

- Under some conditions, we can prove that

$$v(x,t) = U_t(x), \ 0 \le t \le T.$$

$\Rightarrow$ This is why the forward utility is defined as in the previous page!

Musiela and Zariphopoulou (2009, 2010, 2011)

- Musiela and Zariphopoulou (2009, 2010, 2011) develop several examples of correspondence between a forward utility and a dynamic investment strategy.
- They find sufficient conditions for a forward utility to exist and explain the optimality of a dynamic strategy.
- This forward utility is formulated as

$$U_t(x) = u(x, A_t) \tag{2}$$

where $A_t \triangleq \int_0^t \lambda_s^2 ds, t \geq 0$.

$\Rightarrow$ We show how their work can be applied to understand CPPI strategies and life-cycle funds.

Key Idea to find forward utilities

For each strategy $\pi \in \mathcal{A}$, assume that $U_t(X_t^\pi) = u(X_t^\pi, A_t)$. By applying Itô's formula, we have

$$\begin{aligned} dU_t(X_t^\pi) =& u_x(X_t^\pi, A_t)\sigma_t \pi_t dW_t \\ & + \lambda_t^2 \left[u_t(X_t^\pi, A_t) + u_x(X_t^\pi, A_t)\alpha_t + \frac{1}{2} u_{xx}(X_t^\pi, A_t)\alpha_t^2 \right] dt, \end{aligned} \tag{3}$$

where $\alpha_t \triangleq \sigma_t \pi_t / \lambda_t$.

Goal

For each strategy $\pi \in \mathcal{A}$, non-positive drift of $U_t(X_t^\pi)$

$$u_t(X_t^\pi, A_t) + u_x(X_t^\pi, A_t)\alpha_t + \frac{1}{2} u_{xx}(X_t^\pi, A_t)\alpha_t^2 \leq 0$$

For optimal strategy π^*, zero drift of $U_t(X_t^{\pi^*})$

$$u_t(X_t^{\pi^*}, A_t) + u_x(X_t^{\pi^*}, A_t)\alpha_t + \frac{1}{2} u_{xx}(X_t^{\pi^*}, A_t)\alpha_t^2 = 0$$

CPPI Strategy (1)

- Constant Proportion Portfolio Insurance
- Introduced by Black and Perold (1992)
- Key feature : at any time...

$$\text{Value of portfolio} \geq \text{Predefined floor level}$$

- Good way to hedge long-term guarantees when
  - the maturity date is not known in advance
  - regulators require the guarantee to be met at all times
- Popular in the insurance industry to manage pension funds and variable annuities

CPPI Strategy (2)

- $G_t > 0$: predefined floor level. Assume that

$$dG_t = G_t\, r_t\, dt, \; G_0 = G.$$

$$\Rightarrow G_t = GB_t.$$

- V_t: portfolio value at time t
- $C_t = V_t - G_t$: cushion
- Define $X_t = V_t/B_t$, the present value of V_t, then

$$\frac{C_t}{B_t} = X_t - G.$$

- Maintain an exposure to the risky asset S_t proportional to the cushion. (m : multiple)

$$\pi_t = m\,\frac{C_t}{B_t} = m(X_t - G) \tag{4}$$

- The amount of risk-free asset is therefore at all times

$$\pi_t^0 = X_t - \pi_t.$$

Adapted Random Multiple

- To ensure that the CPPI strategy is optimal for an expected utility maximizer at any time horizon in the general market (stochastic parameters), we consider a slightly generalized CPPI strategy with random multiple

$$m_t = \frac{\lambda_t/\lambda_0}{\sigma_t/\sigma_0} m, \ \pi_t = m_t(X_t - G) \tag{5}$$

  At any time t, m_t is adapted to $\mathcal{F}_t$, the information available.
- In the case of a Black-Scholes model (constant parameters), $\pi_t = m_t(X_t - G)$ corresponds to a standard CPPI strategy with fixed multiple m

$$\pi_t = m(X_t - G)$$

  because both λ_t and σ_t are constant.

Proposition 2.1 (General Case)

The dynamic CPPI investment strategy consisting of

$$\pi_t^* = \frac{\lambda_t/\lambda_0}{\sigma_t/\sigma_0} m(X_t^* - G) \tag{6}$$

invested in the risky asset (i.e. a CPPI strategy with an adapted multiple $\frac{\lambda_t/\lambda_0}{\sigma_t/\sigma_0} m$*) corresponds to the optimum for the forward utility* $U_t(x) = u(x, A_t)$ *where* $u(x,s)$ *is given for* $x \in (G, \infty)$ *and* $s \geq 0$ *by*

$$u(x,s) = \begin{cases} \frac{\gamma}{\gamma-1}(x-G)^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}s}, & \gamma \in (0,1) \cup (1,\infty), \\ \ln(x-G) - \frac{s}{2}, & \gamma = 1. \end{cases} \tag{7}$$

where $\gamma = \sigma_0 m/\lambda_0$ *and* $A_t \triangleq \int_0^t \lambda_s^2 ds$*.*

$\Rightarrow$ The forward utility $u(\cdot, s)$ belongs to the HARA utility class at all s.

Proposition 2.2

Reciprocally, given any time T, consider the following portfolio optimization problem to maximize the utility of wealth at time T

$$\max_{\pi \in \mathcal{A}} \mathbb{E}\left[u(X_T, A_T)\right],$$

where $A_T = \int_0^T \lambda_s^2 ds$ and $u(\cdot,\cdot)$ is given by (7) and defined over $(G,\infty) \times [0,\infty)$. Then the optimal allocation is a dynamic CPPI strategy

$$\pi_t^* = \frac{\lambda_t/\lambda_0}{\sigma_t/\sigma_0} m(X_t^* - G).$$

This proposition holds for any given time T with $u(X_T, A_T)$.

$\Rightarrow$ Forward utility: Dynamically consistent utility functions!

We have to rebalance the investment strategy depending on λ_t and σ_t in stochastic environment. (Dynamically changing investment opportunity)

Corollary 2.1 (Black-Scholes Case)

Assume that μ, r and σ are constant and $\lambda \triangleq (\mu - r)/\sigma$. Define $\gamma = \sigma m/\lambda$. Then, we have the following results.

- *With the CPPI strategy $\pi_t^* = m(X_t^* - G)$, the corresponding forward utility is $U_t(x) = u(x, \lambda^2 t)$ with $u(\cdot,\cdot)$ is given by*

$$u(x,s) = \begin{cases} \frac{\gamma}{\gamma-1}(x-G)^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}s}, & \gamma \in (0,1) \cup (1,\infty), \\ \ln(x-G) - \frac{s}{2}, & \gamma = 1. \end{cases} \tag{8}$$

- *Given any time T, the solution to the following portfolio optimization problem*

$$\max_{\pi \in \mathcal{A}} \mathbb{E}[u(X_T, \lambda^2 T)],$$

with $u(\cdot,\cdot)$ given by (8) is a CPPI strategy $\pi_t^* = m(X_t^* - G)$ where the multiple is $m = \frac{\lambda\gamma}{\sigma}$.

Life-Cycle Funds

Key feature of "Life-Cycle Funds"

Investment in risky asset is a decreasing function of time

What we do

- Present the Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA) class of utility functions introduced by Chen, Pelsser, and Vellekoop (2011)
- Give the corresponding forward utility and optimal strategy.
- Show that this optimal strategy displays the age-based investing feature of life-cycle funds which means that the optimal investment in risky asset is a decreasing function of time.

SAHARA Utility Function

- A SAHARA utility function is given by $U(x)$, $x \in \mathbb{R}$, whose absolute risk aversion $\gamma_A(x) = -U_{xx}(x)/U_x(x)$ satisfies

$$\gamma_A(x) = \frac{1}{\sqrt{a^2(x-d)^2 + c^2}}, \tag{9}$$

  with $a > 0$, $c > 0$ and $d \in \mathbb{R}$. When $d = 0$, $U(x)$ is up to a linear transformation, given as follows.
  - If $a = 1$, $U(x) = \frac{1}{2}\ln\left(x + \sqrt{x^2 + c^2}\right) + \frac{1}{2c^2}x\left(\sqrt{x^2+c^2} - x\right)$.
  - If $a \neq 1$, $U(x) = \frac{a(a+1)\left(ax^2 + x\sqrt{a^2x^2+c^2}\right) + c^2}{(a^2-1)\left(ax + \sqrt{a^2x^2+c^2}\right)^{1+\frac{1}{a}}}$.
- For the SAHARA utility: agents may become less risk-averse for very low values of wealth.

Proposition 2.3 (General Case)

The following allocation to risky assets

$$\pi_t^* = \frac{\lambda_t}{\sigma_t}\sqrt{a^2(X_t^*)^2 + b^2 e^{-a^2 A_t}},$$

(where $a > 0$, $b > 0$) is optimal for the forward utility

$$U_t(x) = u(x, A_t)$$

where $u(x,\cdot)$ is a SAHARA utility with time varying parameters, where $A_t = \int_0^t \lambda_s^2 ds$.

π_t^* is also the optimal solution to

$$\max_{\pi\in\mathcal{A}} \mathbb{E}\left[u(X_T, A_T)\right],$$

where u is as in the above proposition.

$\Rightarrow$ Forward utility: Dynamically consistent utility functions!

Corollary 2.2 (Black-Scholes Case)

Assume that μ, r, and σ are constant. The following investment strategy

$$\pi_t^* = \frac{\lambda}{\sigma}\sqrt{a^2(X_t^*)^2 + b^2 e^{-a^2\lambda^2 t}},$$

in the risky asset is optimal for the forward utility $U_t(x) = u(x, \lambda^2 t)$ where $u(x, \cdot)$ is a SAHARA utility as before.
Reciprocally, given any time T, π_t^* also solves

$$\max_{\pi \in \mathcal{A}} \mathbb{E}\left[u(X_T, \lambda^2 T)\right].$$

SAHARA Utility and Life-Cycle Funds

- Local (absolute) risk aversion function, $\gamma(x, s) \triangleq -u_{xx}(x, s)/u_x(x, s)$, in the Black-Scholes model, for the SAHARA utility

$$\gamma(x, s) = \frac{1}{\sqrt{a^2 x^2 + b^2 e^{-a^2 s}}}. \tag{10}$$

- Local risk aversion function (10) is an increasing function of s.
- This means that, if there is an economic agent with a SAHARA utility function, her optimal investment strategy becomes more conservative as time goes.
- As a consequence, the optimal allocation to the risky asset $\pi_t^* = \frac{\lambda}{\sigma}\sqrt{a^2(X_t^*)^2 + b^2 e^{-a^2\lambda^2 t}}$ is a decreasing function of time.

$\Rightarrow$ This is a kind of life-cycle funds!

Stochastic Environment : Rebalancing is Needed

- The optimal strategy in the general case

$$\pi_t^* = \frac{\lambda_t}{\sigma_t}\sqrt{a^2(X_t^*)^2 + b^2 e^{-a^2 A_t}}$$

shares similar features(decreasing in time), but we have to rebalance the investment taking into account λ_t and σ_t because the market is stochastic.

- This is consistent with Viceira (2007) who suggested that the market conditions should be involved in determining the asset allocation path of life-cycle funds.
- The standard life-cycle funds, consisting of a linear decrease of the percentage invested in risky asset does not appear optimal.
- The way to decrease the allocation over time, depends on changes in market conditions and risk aversion.

Conclusion and Future Research Direction

- We studied two popular dynamic investment strategies in the pension funds industry: “CPPI Strategy” and “Life-Cycle Funds”.
- We can conclude that HARA and SAHARA utility functions may play a key role in explaining fund manager’s decisions or in modeling optimal decision making.
- Future research directions include proving the existence and giving an explicit construction of the forward utility for more general investment strategies

Thank you for your attention!

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