Optimal Investment with State-Dependent Constraints

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- This talk is joint work with **Phelim Boyle** (Wilfrid Laurier University, Waterloo, Canada) and with **Steven Vanduffel** (Vrije Universiteit Brussel (VUB), Belgium).

- **Outline of the talk:**

  1. Characterization of optimal investment strategies for an investor with **law-invariant preferences**
  2. Extension to the case when investors have **state-dependent constraints**.

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Part I: Optimal portfolio selection for law-invariant investors

Characterization of optimal investment strategies for an investor with **law-invariant preferences** and a **fixed investment horizon**

- Optimal strategies are "cost-efficient".
- **Cost-efficiency** ⇔ Minimum correlation with the state-price process ⇔ Anti-monotonicity
- In the **Black-Scholes** setting,
  - *Optimality* of strategies increasing in S_T.
  - *Suboptimality* of path-dependent contracts.

What is "cost-efficiency"?

Cost-Efficiency

A strategy (or a payoff) is **cost-efficient** if any other strategy that generates the same distribution under P costs at least as much.

This concept was originally proposed by Dybvig (1988).

Main Assumptions

- Consider an arbitrage-free and complete market.
- Given a strategy with final payoff X_T at time T. There exists a unique probability measure Q, such that its price at 0 is

$$c(X_T) = \mathbb{E}_Q[e^{-rT} X_T]$$

Distributional price of a cdf F under the physical measure P.

$$PD(F) = \min_{\{Y \mid Y \sim F\}} c(Y)$$

- The strategy with payoff X_T is **cost-efficient** if

$$PD(F) = c(X_T)$$

Traditional Approach to Portfolio Selection

Consider an investor with **increasing law-invariant** preferences and a **fixed** horizon. Denote by X_T the investor's final wealth.

- Optimize an increasing law-invariant objective function
  1. $\max_{\mathbf{X_T}} (\mathbf{E_P}[\mathbf{U}(\mathbf{X_T})])$ where U is increasing.
  2. Minimizing Value-at-Risk (a quantile of the cdf)
  3. Probability target maximizing: $\max_{\mathbf{X_T}} \mathbf{P}(\mathbf{X_T} > \mathbf{K})$
  4. ...
- for a given **cost** (budget) *cost at* $0 = E_Q[e^{-rT} X_T]$.

Find optimal strategy X_T^* $\Rightarrow$ Optimal cdf F of X_T^*

It is clear that the optimal strategy must be **cost-efficient**

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Assumptions

To characterize cost-efficiency, we need to introduce the "state-price process"

- Given a payoff X_T at time T. P ("physical measure") and Q ("risk-neutral measure") satisfy

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T, \quad \mathbf{c(X_T)} = \mathbb{E}_Q[e^{-rT}X_T] = \mathbb{E}_{\mathbf{P}}[\xi_{\mathbf{T}}\mathbf{X_T}].$$

ξ_T is called "state-price process".

Theorem (Sufficient condition for cost-efficiency)

Any random payoff X_T with the property that (X_T, ξ_T) is ***anti-monotonic*** *is* ***cost-efficient****.*

X_T and ξ_T are **anti-monotonic**: "When ξ_T increases, then X_T decreases".

Idea of the proof

Minimizing the price $c(X_T) = E[\xi_T X_T]$ when $X_T \sim F$ amounts to finding the dependence structure that **minimizes the correlation** between the strategy and the state-price process

$$\min_{X_T} \ \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

Recall that

$$\text{corr}(X_T, \xi_T) = \frac{\mathbb{E}[\xi_T X_T] - \mathbb{E}[\xi_T]\mathbb{E}[X_T]}{\text{std}(\xi_T)\,\text{std}(X_T)}.$$

When the distributions for both X_T and ξ_T are fixed, we have

$$(X_T, \xi_T) \text{ is anti-monotonic} \Rightarrow \text{corr}[X_T, \xi_T] \text{ is minimal.}$$

Explicit Representation for Cost-efficiency

Assume ξ_T is continuously distributed (for example a Black-Scholes market)

Theorem (Necessary and sufficient Condition)

The cheapest strategy that has cdf F is given explicitly by

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

Note that $X_T^\star \sim F$ and $X_T^\star$ is a.s. **unique** *such that*

$$PD(F) = c(X_T^\star) = \mathbb{E}[\xi_T X_T^\star]$$

where F^{-1} is defined as follows:

$$F^{-1}(y) = \min\left\{x \;/\; F(x) \geqslant y\right\}.$$

Idea of the proof

Solving this problem amounts to finding bounds on copulas!

$$\min_{X_T} \ \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

The distribution G is known and depends on the financial market.
Let C denote a copula for (ξ_T, X).

$$\mathbb{E}[\xi_T X] = \int\int (1 - G(\xi) - F(x) + C(G(\xi), F(x)))dxd\xi, \quad (1)$$

Bounds for $\mathbb{E}[\xi_T X]$ are derived from bounds on C

$$\max(\mathbf{u} + \mathbf{v} - \mathbf{1}, \mathbf{0}) \leqslant C(u, v) \leqslant \min(u, v)$$

(Fréchet-Hoeffding Bounds for copulas) (**anti-monotonic** copula)

Black-Scholes Model

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Then

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right) = a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = e^{\frac{\theta}{\sigma}(\mu - \frac{\sigma^2}{2})t - (r + \frac{\theta^2}{2})t}$ and $b = \frac{\mu - r}{\sigma^2}$.

Theorem (Cost-efficiency in Black-Scholes model)

To be cost-efficient, the contract has to be a European derivative written on S_T and non-decreasing w.r.t. S_T (when $\mu > r$). In this case,

$$\mathbf{X_T^\star = F^{-1}\left(F_{S_T}\left(S_T\right)\right)}$$

Geometric Asian contract in Black-Scholes model

Assume a strike K. The payoff of the Geometric Asian call is given by

$$X_T = \left(e^{\frac{1}{T}\int_0^T \ln(S_t)dt} - K\right)^+$$

which corresponds in the discrete case to $\left(\left(\prod_{k=1}^{n} S_{\frac{kT}{n}}\right)^{\frac{1}{n}} - K\right)^+$.

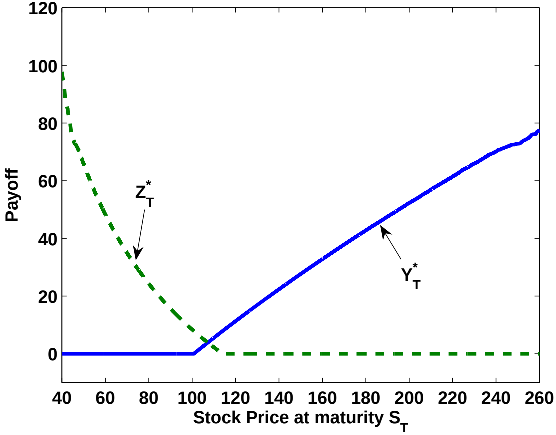
The efficient payoff that is distributed as the payoff X_T is a power call option

$$X_T^\star = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

where $d := S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$.

Similar result in the discrete case.

Example: Discrete Geometric Option



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

$$C(X_T^\star) = 5.3 < Price(geometric\ Asian) = 5.5 < C(Z_T^\star) = 8.4.$$

Put option in Black-Scholes model

Assume a strike K. The payoff of the put is given by

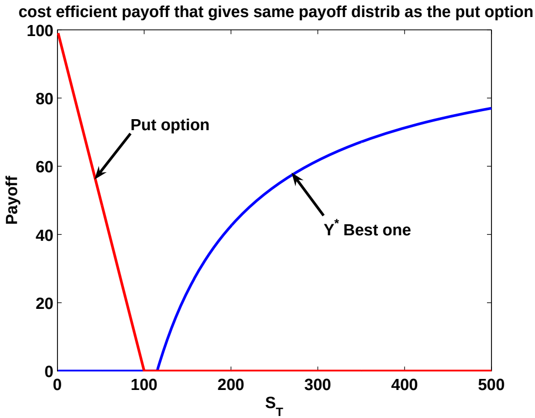
$$L_T = (K - S_T)^+ .$$

The payout that has the **lowest** cost and that has the same distribution as the put option payoff is given by

$$Y_T^\star = F_L^{-1}\left(F_{S_T}\left(S_T\right)\right) = \left(K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T}\right)^+ .$$

This type of power option "dominates" the put option.

Cost-efficient payoff of a put



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional price of the put $= 3.14$

Price of the put $= 5.57$

Efficiency loss for the put $=$ 5.57-3.14$=$ 2.43

Explaining the Demand for Inefficient Payoffs

1. **Other sources of uncertainty:** Stochastic interest rates or stochastic volatility
2. **Transaction costs, frictions**
3. **Intermediary consumption.**
4. Often we are looking at an **isolated contract**: the theory applies to the complete portfolio.
5. **State-dependent needs**
   - **Background risk:**
     - Hedging a long position in the market index S_T (background risk) by purchasing a put option,
     - the background risk can be path-dependent.
   - **Stochastic benchmark or other constraints:** If the investor wants to outperform a given (stochastic) benchmark Γ such that:

$$P\left\{\omega \in \Omega \,/\, W_T(\omega) > \Gamma(\omega)\right\} \geqslant \alpha.$$

Part 2:
Investment with State-Dependent Constraints

Problem considered so far

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\xi_T X_T\right].$$

A payoff that solves this problem is **cost-efficient**.

New Problem

$$\min_{\{Y_T \mid Y_T \sim F, \mathbb{S}\}} \mathbb{E}\left[\xi_T Y_T\right].$$

where $\mathbb{S}$ denotes a set of constraints. A payoff that solves this problem is called a $\mathbb{S}-$**constrained cost-efficient payoff**.

How to formulate "state-dependent constraints"?

Y_T and S_T have given distributions.

- The investor wants to ensure a **minimum** when the market falls

$$\mathbb{P}(Y_T > 100 \mid S_T < 95) = 0.8.$$

  This provides some additional information on the joint distribution between Y_T and $S_T \Rightarrow$ information on the joint distribution of (ξ_T, Y_T) in the Black-Scholes framework.

- Y_T is **decreasing** in S_T when the stock S_T falls below some level (to justify the demand of a put option).
- Y_T is **independent** of S_T when S_T falls below some level.

All these constraints impose the strategy Y_T to pay out in given states of the world.

Formally

Goal: Find the **cheapest** possible payoff Y_T with the distribution F and which **satisfies additional constraints** of the form

$$\mathbb{P}(\xi_T \leqslant x, Y_T \leqslant y) = Q(F_{\xi_T}(x), F(y)),$$

with $x > 0, y \in \mathbb{R}$ and Q a given feasible function (for example a copula).

Each constraint gives information on the dependence between the state-price ξ_T and Y_T and is, for a given function Q, determined by the pair $(F_{\xi_T}(x), F(y))$.

Denote the finite or infinite set of all such constraints by $\mathbb{S}$.

Sufficient condition for the existence

Theorem

Let $t \in (0, T)$. If there exists a copula L satisfying $\mathbb{S}$ such that **L ⩽ C** *(pointwise) for all other copulas C satisfying $\mathbb{S}$ then the payoff $Y_T^\star$ given by*

$$Y_T^\star = F^{-1}(f(\xi_T, \xi_t))$$

is a $\mathbb{S}$-constrained cost-efficient payoff. Here $f(\xi_T, \xi_t)$ is given by

$$f(\xi_T, \xi_t) = \left(\ell_{F_{\xi_T}(\xi_T)}\right)^{-1}\left[j_{F_{\xi_T}(\xi_T)}(F_{\xi_t}(\xi_t))\right],$$

where the functions $j_u(v)$ and $\ell_u(v)$ are defined as the first partial derivative for $(u, v) \to J(u, v)$ and $(u, v) \to L(u, v)$ respectively and where J denotes the copula for the random pair (ξ_T, ξ_t).

If (U, V) has a copula L then $\ell_u(v) = \mathbb{P}(V \leqslant v | U = u)$.

Example 1: $\mathbb{S} = \emptyset$ (no constraints)

From the Fréchet-Hoeffding bounds on copulas one has

$$\forall (u,v) \in [0,1]^2, \quad C(u,v) \geqslant \max(0,\ u+v-1).$$

Note that $L(u,v) := \max(0,\ u+v-1)$ is the anti-monotonic copula.
Then one obtains $\ell_u(v) = 1$ if $v > 1-u$ and that $\ell_u(v) = 0$ if $v < 1-u$. Hence we find that $\ell_u^{-1}(p) = 1-u$ for all $0 < p \leqslant 1$ which implies that

$$f(\xi_t, \xi_T) = 1 - F_{\xi_T}(\xi_T).$$

It follows that $Y_T^\star$ is given by

$$\mathbf{Y}_\mathbf{T}^\star = \mathbf{F}^{-1}\left(\mathbf{1} - \left(\mathbf{F}_{\xi_\mathbf{T}}\left(\xi_\mathbf{T}\right)\right)\right)$$

Existence of the optimum ⇔ Existence of minimum copula

Theorem (Sufficient condition for existence of a minimal copula L)

Let $\mathbb{S}$ be an increasing and compact subset of $[0,1]^2$. Then a minimal copula $L(u,v)$ satisfying $\mathbb{S}$ exists and is given by

$$L(u,v) = \max\{0,\ u+v-1,\ K(u,v)\}.$$

where $K(u,v) = \max_{(a,b)\in\mathbb{S}}\{Q(a,b) - (a-u)^+ - (b-v)^+\}$.

Proof in Tankov (2011, Journal of Applied Probability).

Consequently the existence of a $\mathbb{S}-$constrained cost-efficient payoff is guaranteed when $\mathbb{S}$ is increasing and compact.

Example 2: $\mathbb{S}$ contains 1 constraint

Assume a Black-Scholes market. We suppose that the investor is looking for the payoff Y_T such that $Y_T \sim F$ (where F is the cdf of S_T) and satisfies the following constraint

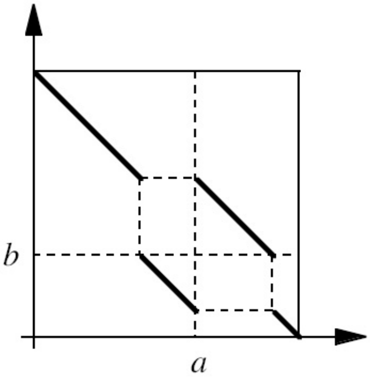
$$\mathbb{P}(S_T < 95,\ Y_T > 100) = 0.2.$$

The optimal strategy, where $a = 1 - F_{S_T}(95)$, $b = F_{S_T}(100)$ and $\vartheta = 0.2 - F_{S_T}(95) + F_{S_T}(100)$ is given by the previous theorem.

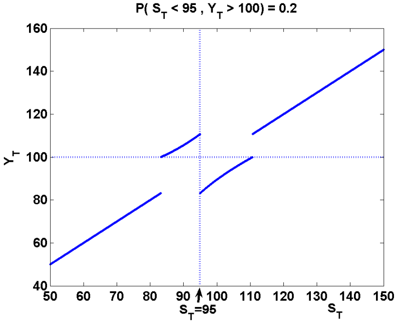
Its price is 100.2

Example 2: Illustration

Minimum Copula



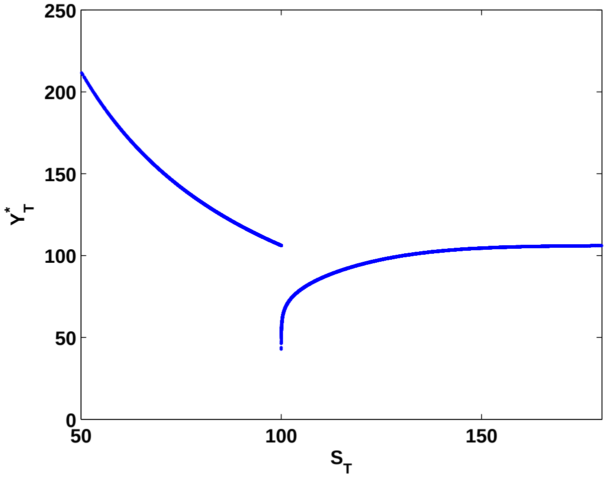
Optimal Strategy



Example 3: $\mathbb{S}$ is infinite

A cost-efficient strategy with the same distribution F as S_T but such that it is decreasing in S_T when $S_T \leqslant \ell$ is unique a.s. The **constrained cost-efficient payoff** can be written as

$$Y_T^{\star} := F^{-1}\left[(1 - F(S_T))\mathbb{1}_{S_T<\ell} + \left(F(S_T) - F(\ell)\right)\mathbb{1}_{S_T\geqslant\ell}\right].$$



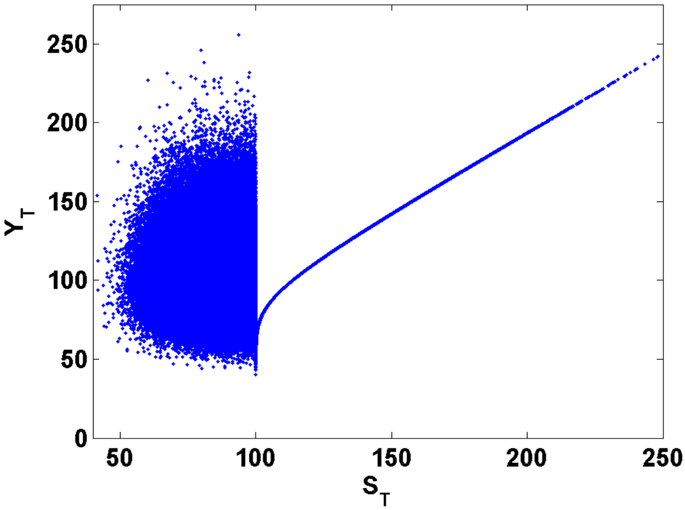
$Y_T^\star$ as a function of S_T. Parameters: $\ell = 100$, $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$ and $r = 0.03$. The price is 103.4.

Example 4: $\mathbb{S}$ is infinite

A cost-efficient strategy with the same distribution F as S_T but such that it is independent of S_T when $S_T \leqslant \ell$ can be constructed as

$$Y_T^\star = F^{-1}\left(\Phi\left(k(S_t, S_T)\right)\mathbb{1}_{S_T<\ell} + \left(\frac{F(S_T)-F(\ell)}{1-F(\ell)}\right)\mathbb{1}_{S_T\geqslant\ell}\right),$$

where $k(S_t, S_T) = \frac{\ln\left(\frac{S_t}{S_T^{t/T}}\right) - (1-\frac{t}{T})\ln(S_0)}{\sigma\sqrt{t-\frac{t^2}{T}}}$ and $t \in (0, T)$ can be chosen freely (**Not unique! and path-dependent optimum!**).



10,000 realizations of $Y_T^\star$ as a function of S_T where $\ell = 100$, $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$, $r = 0.03$ and $t = T/2$. Its price is 101.1

Conclusion

- Characterization of cost-efficient strategies.
- **Path-dependent strategies are never optimal in the Black and Scholes model** for investors with law-invariant preferences.
- Optimal investment choice under state-dependent constraints. **In the presence of state-dependent constraints, optimal strategies**
  - **are not always non-decreasing with the stock price S_T.**
  - **are not anymore unique and could be path-dependent.**

Further Research Directions / Work in Progress (1/2)

- Extension to the presence of **stochastic interest rates** and application to executive compensation (work in progress with Jit Seng Chen and Phelim Boyle).
- Extension to the case when there is uncertainty on the state-price process (incompleteness of the market).
- Extension to the case when there is uncertainty on the cdf F (joint work with Steven Vanduffel).

Further Research Directions / Work in Progress (2/2)

- Using cost-efficiency to derive **bounds for insurance prices** derived from indifference utility pricing (working paper on "Bounds for Insurance Prices" with Steven Vanduffel) and more generally application to utility indifference pricing in incomplete market.
- Further extend the work on state-dependent constraints:
  1. Solve with **expectations constraints** between ξ_T and X_T.

     $$\mathbb{E}[g_i(\xi_T, X_T)] \in I_i$$

     where I_i is an interval, possibly reduced to a single value.
  2. Solve with the probability constraint of outperforming a benchmark

     $$\mathbb{P}(X_T > h(S_T)) \geqslant \varepsilon$$

  3. Extend the literature on optimal portfolio selection in specific models under state-dependent constraints.

Do not hesitate to contact me to get updated working papers!

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