Financial Bounds for Insurance Claims

Carole Bernard (University of Waterloo, WatRISQ)

Steven Vanduffel (Vrije Universiteit Brussel, Belgium).

Background & Objectives

- (*"Explicit representation of Cost-efficient Strategies"* with Phelim Boyle (Wilfrid Laurier University))

- Given a cdf F, there exists an explicit representation of $X_T^\star$ and of $Z_T^\star$ such that
  - $X_T^\star \sim F$ and $Z_T^\star \sim F$ in the real world
  - $X_T^\star$ is the cheapest strategy (= cost-efficient strategy)
  - $Z_T^\star$ is the most expensive strategy (= cost-inefficient strategy)

  $\Rightarrow$ **Price(claim)**$\in \left[c(X_T^\star), c(Z_T^\star)\right]$

- Our objectives:
  1. To propose a "market-consistent" pricing tool
  2. To find similar bounds
     - on prices of claims that cannot be hedged perfectly in the market.
     - but for which we know the cdf under the physical probability.

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     - but for which we know the cdf under the physical probability.

Some Assumptions

- Consider an arbitrage-free and complete market.
- Given a strategy with payoff X_T at time T. There exists Q, such that its price at 0 is

$$P_X = \mathbb{E}_Q[e^{-rT} X_T]$$

- P ("physical measure") and Q ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT} \left(\frac{dQ}{dP}\right)_T, \quad \mathbf{c(X_T)} = \mathbb{E}_Q[e^{-rT} X_T] = \mathbb{E}_\mathbf{P}[\xi_\mathbf{T} \mathbf{X_T}].$$

- Given a strategy with payoff X_T at time T, and initial price at time 0

$$c(X) = \mathbb{E}\left[\xi_T X_T\right]$$

- F : X_T's distribution under the **physical measure** P.

The distributional price is defined as

$$PD(F) = \min_{\{Y \mid Y \sim F\}} \{\mathbb{E}\left[\xi_T Y\right]\} = \min_{\{Y \mid Y \sim F\}} c(Y)$$

(lower bound on the price of a financial claim with cdf F)

$\Rightarrow$ Example of $X \sim Y$ with different costs in a binomial tree.

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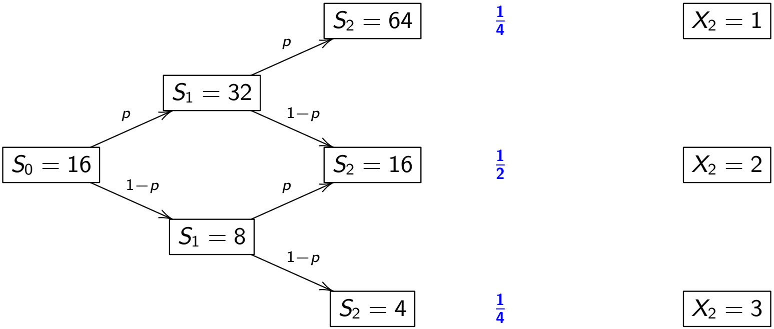
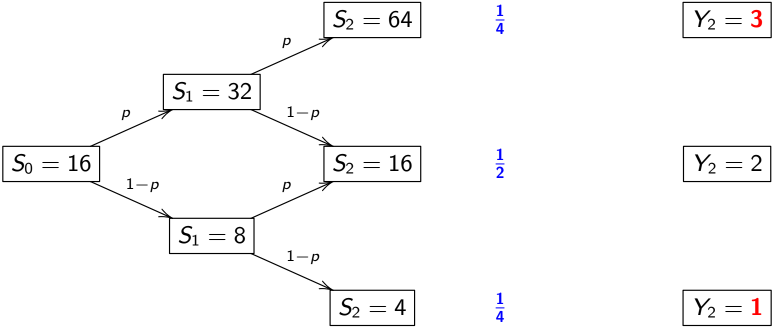
A simple illustration for X_2, a payoff at $T = 2$

Real-world probabilities$=p = \frac{1}{2}$



Y_2, a payoff at $T = 2$ distributed as X_2

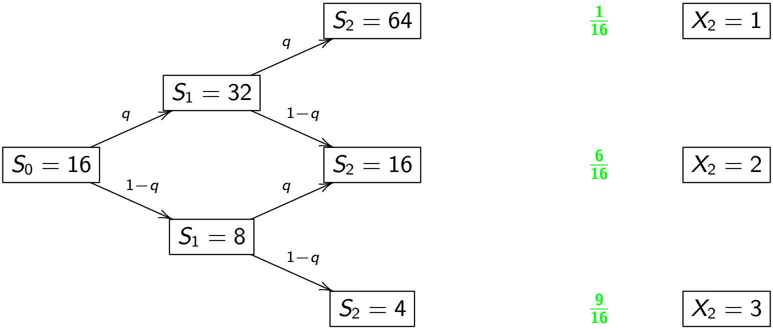
Real-world probabilities=$p = \frac{1}{2}$



X and Y have the same distribution under the physical measure

X_2, a payoff at $T = 2$

risk neutral probabilities$=q = \frac{1}{4}$.

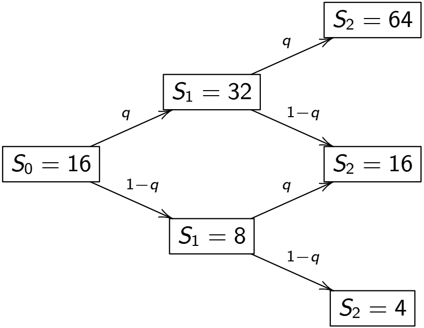


$$P_{X_2} = \text{Price of } X_2 = \left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2}$$

Y_2, a payoff at $T = 2$

risk neutral

probabilities$=q = \frac{1}{4}$.



$\frac{1}{16}$ $\mathbf{Y_2 = 3}$

$\frac{6}{16}$ $Y_2 = 2$

$\frac{9}{16}$ $\mathbf{Y_2 = 1}$

$$P_{Y_2} = \left(\frac{1}{16}3 + \frac{6}{16}2 + \frac{9}{16}1\right) = \frac{3}{2}$$

$$P_{X_2} = \text{Price of } X_2 = \left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2} \quad ,$$

Minimum Price = Cost-efficient Strategy

Theorem

Consider the following optimization problem:

$$\min_{\{Z \mid Z \sim F\}} \{\mathbb{E}[\xi_T Z]\}$$

Assume ξ_T is continuously distributed, then the optimal strategy is

$$X_T^\star = F^{-1}(1 - F_\xi(\xi_T)).$$

Note that $X_T^\star \sim F$ and $X_T^\star$ is a.s. unique such that

$$PD(F) = c(X_T^\star) = \mathbb{E}\left[\xi_T X_T^\star\right]$$

Maximum price = Least Efficient Strategy

Theorem

Consider the following optimization problem:

$$\max_{\{Z \mid Z \sim F\}} \{\mathbb{E}[\xi_T Z]\}$$

Assume ξ_T is continuously distributed. The strategy $Z_T^\star$ that generates the same distribution as F with the highest cost can be described as follows:

$$Z_T^\star = F^{-1}(F_\xi(\xi_T))$$

Black and Scholes Model

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Then

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right) = a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = e^{\frac{\theta}{\sigma}(\mu-\frac{\sigma^2}{2})t-(r+\frac{\theta^2}{2})t}$ and $b = \frac{\mu-r}{\sigma^2}$.
To be cost-efficient, the contract has to be a European derivative written on S_T and non-decreasing w.r.t. S_T (when $\mu \geqslant r$). In this case,

$$X_T^\star = F^{-1}\left(F_{S_T}\left(S_T\right)\right)$$

Geometric Asian contract in Black and Scholes model

Assume a strike K. The payoff of the Geometric Asian call is given by

$$X_T = \left(e^{\frac{1}{T}\int_0^T \ln(S_t)dt} - K\right)^+$$

which corresponds in the discrete case to $\left(\left(\prod_{k=1}^n S_{\frac{kT}{n}}\right)^{\frac{1}{n}} - K\right)^+$.

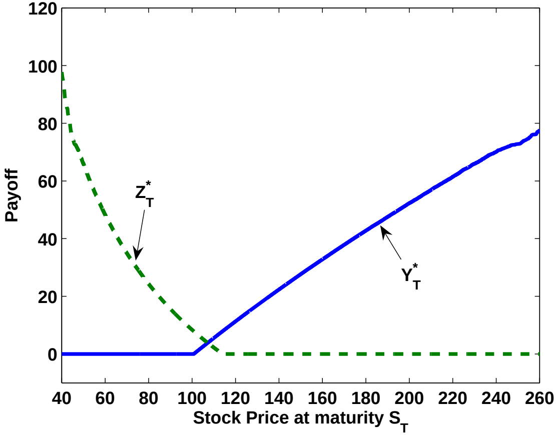
The efficient payoff that is distributed as the payoff X_T is a power call option

$$X_T^\star = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

where $d := S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$.

Similar result in the discrete case.

Example: Discrete Geometric Option



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100,\ n = 12.$

$$C(X_T^\star) = 5.77 < Price(geometric\ Asian) = 5.94 < C(Z_T^\star) = 9.03.$$

Bounds on Prices

- Consider a **financial claim** at time T with cdf F.
- Denote by $X_T^\star$ the cheapest strategy with cdf F and by $Z_T^\star$ the most expensive strategy with cdf F,

$$\Rightarrow \textbf{Cost(claim)} \in \left[c(X_T^\star), c(Z_T^\star)\right]$$

How to use these bounds for insurance claims?

1. Let C_T be a random non-negative **insurance payoff** (not traded) with distribution F.
2. Under some conditions, it also follows that

$$Price(C_T) \geqslant c(X_T^\star).$$

but in general there is no upper bound (independent of the preferences).

Assumptions on Preferences

Denote by X_T the final wealth of the investor and $V(X_T)$ the objective function of the agent.

1. Market participants all have a fixed investment horizon $T > 0$ and there is no intermediate consumption (one-period model).
2. **Agents' preferences depend only on the probability distribution of terminal wealth:** "law-invariant" preferences. (if $X_T \sim Z_T$ then: $V(X_T) = V(Z_T)$.)
3. **Agents prefer "more to less"**: if c is a non-negative random variable $V(X_T + c) \geqslant V(X_T)$.
4. **Agents are risk-averse**:

$$\begin{cases} E[X_T] = E[Y_T] \\ \forall d \in \mathbb{R}, E[(X_T - d)^+] \leq E[(Y_T - d)^+] \end{cases} \Rightarrow V(X_T) \geqslant V(Y_T)$$

Bid and Ask prices for insurance claims in the absence of a financial market using "certainty equivalents"

- From the **viewpoint of the insured** with objective function $U(\cdot)$ and initial wealth ω the (bid) price, p^b,

$$U[(\omega - p^b)e^{rT}] = U[\omega e^{rT} - C_T].$$

- From the **viewpoint of the insurer** with a given objective function $V(\cdot)$ and initial wealth ω the ask price, p^a,

$$V[(\omega + p^a)e^{rT} - C_T] = V[\omega e^{rT}].$$

Properties

1. Bid and Ask prices verify

$$p_\bullet \geqslant e^{-rT}\mathbb{E}[C_T].$$

2. If the insurer is risk neutral ($v(x) = x$), then

$$p_b \geqslant p_a = e^{-rT}\mathbb{E}[C_T]$$

3. In the case of exponential utility $p_a = p_b$.
4. In the case of Yaari's theory $p_a = p_b$.
5. In general, nothing can be said. $u(x) = v(x) = 1 - 1/x$, both agents have same initial wealth, $C_T \sim U(0, 2)$. Next figure

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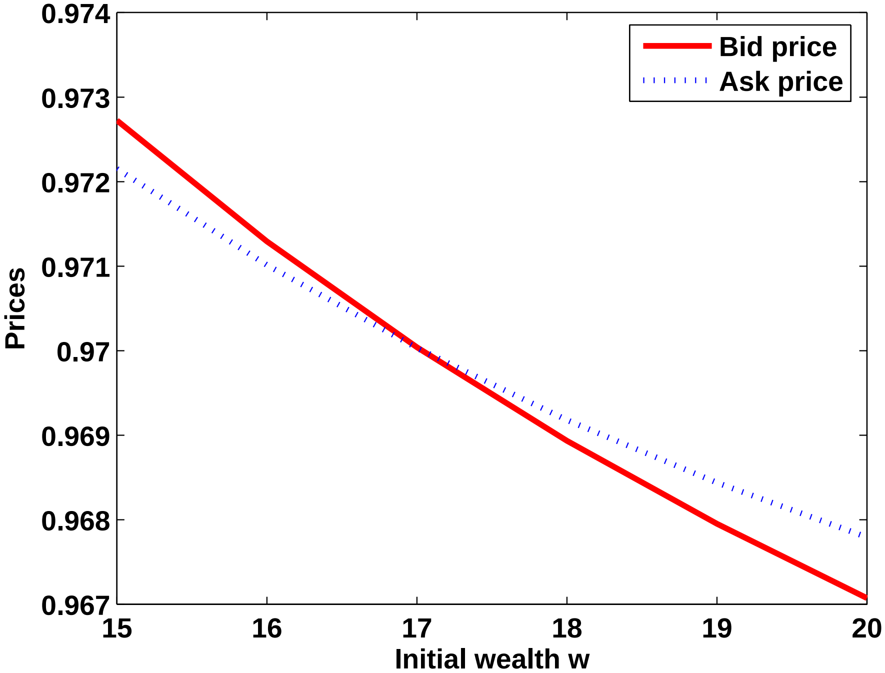

Bid price
Ask price
Prices
0.974
0.973
0.972
0.971
0.97
0.969
0.968
0.967
15
16
17
18
19
20
Initial wealth w

Bid and Ask prices in the presence of a financial market

• From the **viewpoint of the insured** with objective $U(\cdot)$ and initial wealth ω the (bid) price, p^b, follows from

$$\sup_{X_T \in A(\omega - p^b)} \{U[X_T]\} = \sup_{X_T \in A(\omega)} \{U[X_T - C_T]\}.$$

• From the **viewpoint of the insurer** with objective $V(\cdot)$ and initial wealth ω the ask price, p^a, follows from

$$\sup_{X_T \in A(\omega + p^a)} \{V[X_T - C_T]\} = \sup_{X_T \in A(\omega)} \{V[X_T]\}.$$

• In general computing explicitly p^b and p^a is not in reach.
• (Market Consistency) If C_T is hedgeable, then

$$p_b = p_a = \mathbb{E}[\xi_T C_T].$$

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Lower bound

- Assuming that decision makers are risk averse,

Theorem

Using the abusive notation $p^\bullet$ to reflect both p^a and p^b,

$$p^\bullet \geq \mathbb{E}[\xi_T.C_T].$$

Furthermore, the lower bound $\mathbb{E}[\xi_T.C_T]$ is the market price of the financial payoff $\mathbb{E}[C_T|\xi_T]$

- Note that

$$p^\bullet \geq e^{-rT}.\mathbb{E}[C_T] + Cov[C_T, \xi_T].$$

Comments

• Hence when the claim C_T and the state-price ξ_T are **negatively** correlated we find that $e^{-rT}.\mathbb{E}[C_T]$ **is no longer a lower bound** for p^b and p^a which contrasts with traditional bound stated in many actuarial textbooks on insurance pricing.

• Finally, remark that the inequality essentially states that both the insured and the insurer are prepared to agree on a price for the **insurance payoff** C_T which is larger than the price "as if C_T would be a **financial payoff"**.

Comments (Cont'd): 3 cases:

- C_T **is independent of the market**,

$$p^\bullet \geq e^{-rT}.\mathbb{E}[C_T].$$

- C_T **is positively correlated with the state-price process**, the classical lower bound $e^{-rT}\mathbb{E}[C_T]$ is now strictly improved.

$$p^\bullet \geq e^{-rT}.\mathbb{E}[C_T] + Cov[C_T, \xi_T] > e^{-rT}.\mathbb{E}[C_T].$$

- C_T **is negatively correlated with the state-price process**, the lower bound is smaller

$$p^\bullet \geq e^{-rT}.\mathbb{E}[C_T] + Cov[C_T, \xi_T].$$

The best lower bound for equity-linked insurance benefits will generally be lower than $e^{-rT}\mathbb{E}[C_T]$ because

$$Cov(S_T, \xi_T) = \mathbb{E}[S_T\xi_T] - \mathbb{E}[S_T]\mathbb{E}[\xi_T] = e^{-rT}(\mathbb{E}_{\mathbb{Q}}[S_T] - \mathbb{E}[S_T]),$$

Index-Linked Contract

- A life insurance company wants to reinsure payments of $(K - S_T)^+$ paid to a policyholder if alive at time T.

$$C_T = (K - S_T)^+ \mathbb{1}_{\tau > T}$$

  where τ denotes the policyholder's time of death.

- A reinsurer offers full coverage.

$$\mathbb{E}[\xi_T \mathbb{E}[C_T | \xi_T]] = \mathbb{E}[\xi_T C_T] = p(e^{-rT} K - S_0 + C_{bs}(S_0, K, T))$$

  where $p = \mathbb{P}(\tau > T)$ and $C_{bs}(S_0, K, T)$ is the Black Scholes call price.

- u: insurer's utility

$$u(x) = 1 - \frac{\exp(-\gamma x)}{\gamma}.$$

  where the absolute risk aversions $\gamma > 0$.

Bid Price

Define $k_1(.)$ and $k_2(.)$ such that for a given wealth z

$$k_1(z) = \sup_{X_T \in A(z)} \mathbb{E}\left[u\left(X_T - C_T\right)\right]$$

and

$$k_2(z) = \sup_{X_T \in A(z)} \mathbb{E}\left[u\left(X_T\right)\right].$$

To calculate explicitly $k_1(z)$, we first observe that

$$\begin{aligned} k_1(z) &= \sup_{X_T \in A(z)} \mathbb{E}\left[\mathbb{E}\left[u\left(X_T - (K - S_T)^+ \mathbb{1}_{\tau > T}\right) | \tau\right]\right] \\ &= \sup_{X_T \in A(z)} \mathbb{E}\left[pu\left(X_T - (K - S_T)^+\right) + (1-p)u\left(X_T\right)\right] \end{aligned}$$

where $p = \mathbb{P}(\tau > T)$ and τ is independent of X_T and S_T.

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where $p = \mathbb{P}(\tau > T)$ and τ is independent of X_T and S_T.

Computation of k_1: Pathwise Optimization

We maximize pathwise. Let $\omega \in \Omega$, then define

$$\phi(x) = pu\left(x - (K - S_T(\omega))^+\right) + (1-p)u\left(x\right) - \lambda\xi_T(\omega)x$$

It is obvious that $\phi'' \leqslant 0$ and therefore that ϕ is concave and attains its maximum at x^* defined by

$$\phi'(x^*) = 0.$$

For $\lambda > 0$ and for each $\omega \in \Omega$, define $X_T^*(\lambda, \omega) = x^*$. If there exists λ such that $\mathbb{E}[\xi_T X_T^*(\lambda)] = z$ then $X_T^*(\lambda)$ is an optimal solution and

$$k_1(z) = \mathbb{E}[u\left(X_T^* - (K - S_T)^+ \mathbb{1}_{\tau > T}\right)].$$

Illustration

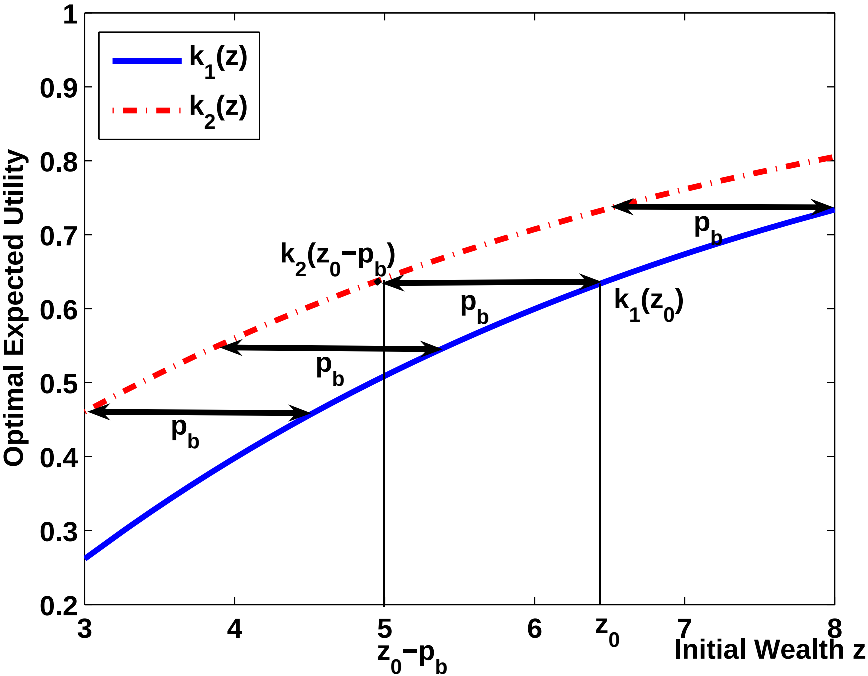
Next slide illustrates how to calculate explicitly bid prices.
Recall that for a given wealth z

$$k_1(z) = \sup_{X_T \in A(z)} \mathbb{E}\left[u\left(X_T - C_T\right)\right]$$

and

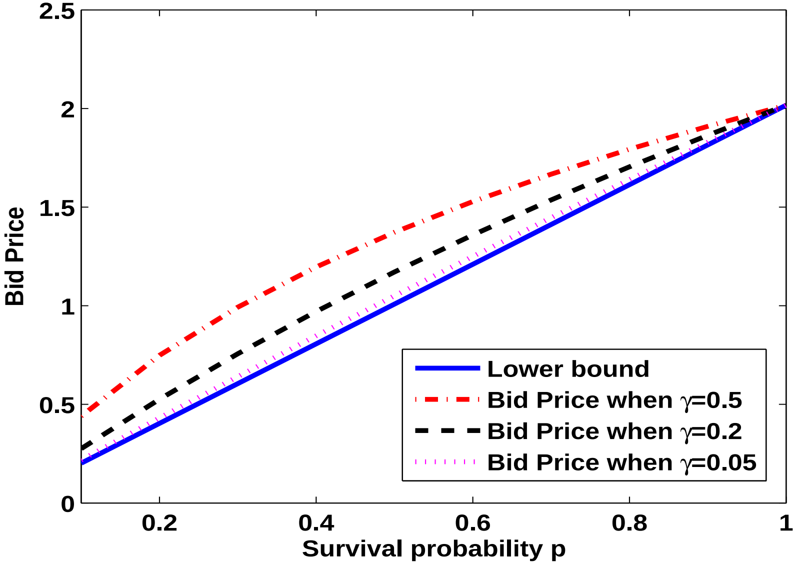
$$k_2(z) = \sup_{X_T \in A(z)} \mathbb{E}\left[u\left(X_T\right)\right].$$

Parameters are $r = 2\%$, $\sigma = 0.2$, $\mu = 4\%$, $S_0 = 10$, $T = 1$, $K = 12$, $\gamma = 0.2$, $p = 0.7$.



k_1(z)
k_2(z)
Optimal Expected Utility
Initial Wealth z
k_2(z_0−p_b)
k_1(z_0)
p_b
p_b
p_b
p_b
z_0
z_0−p_b
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
3
4
5
6
7
8

Bid and ask prices with respect to survival probability p



Conclusion

- Market consistent pricing of insurance claims
- Preference-free bounds on prices of financial and insurance claims
- These bounds correspond to prices of some financial payoffs that we give explicitly
- These bounds are robust in the sense that they are derived under rather mild assumptions

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Thanks!

Additional Material

Put option in Black and Scholes model

Assume a strike K. The payoff of the put is given by

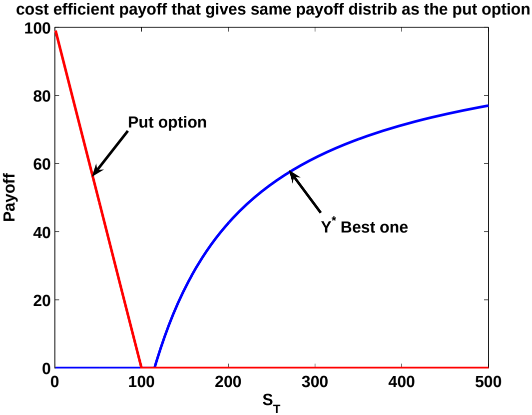
$$L_T = (K - S_T)^+ .$$

The payoff that has the **lowest** cost and is distributed such as the put option is given by

$$Y_T^\star = F_L^{-1}\left(F_{S_T}(S_T)\right) = \left(K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T}\right)^+ .$$

This type of power option “dominates” the put option.

Cost-efficient payoff of a put



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional price of the put = 3.14

Price of the put = 5.57

Efficiency loss for the put = 5.57-3.14= 2.43

Toy example Equity-Linked Insurance

Simplest possible insurance claim that pays at time $T = 1$ a payoff C_1 distributed as a Bernoulli with parameter $p = 0.001$.
$\mathbb{P}(C_1 = 1) = p$ and $\mathbb{P}(C_1 = 0) = 1 - p$.

3 cases:

First, the insurance claim C is linked to the death of a specific individual, then

$$\mathbb{E}[C_1|\xi_1] = \mathbb{E}[C_1].$$

Bid and ask prices $p^\bullet$ satisfy

$$p^\bullet \geq \mathbb{E}[\xi_1 \mathbb{E}[C_1|\xi_1]] = e^{-r}\mathbb{E}[C_1] = e^{-r}\mathbb{P}(death).$$

Second, C_1 pays 1 if a designated person dies and the risky asset in the market is higher than a value H or equivalently $\{\xi_1 < L\} = \{S_1 > H\}$) and

$$\begin{aligned} \mathbb{E}[C_1|\xi_1] &= \mathbb{E}[\mathbb{1}_{death}\mathbb{1}_{\xi_1<L}|\xi_1] \\ &= \mathbb{P}(death)\mathbb{1}_{\xi_1<L}. \end{aligned}$$

The market price of the claim $\mathbb{E}[C_1|\xi_1]$ is $e^{-r}.\mathbb{P}(death)\mathbb{Q}(S_1 > H)$ and thus bid and ask prices satisfy

$$p^{\bullet} \geq e^{-r}.\mathbb{P}(death)\mathbb{Q}(S_1 > H),$$

$e^{-r}\mathbb{E}[C_1] = e^{-r}\mathbb{P}(death)\mathbb{P}(S_1 > H) > e^{-r}\mathbb{P}(death)\mathbb{Q}(S_1 > H)$.
Third, C_1 pays 1 if a designated person dies and the risky asset in the market is lower than a value H. Then, $Cov(C_1, \xi_1) > 0$ and

$$p^{\bullet} \geq \mathbb{E}[\xi_1\mathbb{E}[C_1|\xi_1]] = \mathbb{P}(death).\mathbb{Q}(S_1 < H) > e^{-r}\mathbb{E}[C_1].$$

Corollary: Optimal Investment (key)

Corollary

Denote by $V(\cdot)$ the objective function and given an initial wealth $w \in \mathbb{R}^+$ it holds that

$$\sup_{X_T \in A(w)} V(X_T) = \sup_{X_T \in A_\xi(w)} V(X_T), \tag{1}$$

where

- *$A(w)$ is the set of random wealths X_T that can be generated at maturity $T > 0$ with an initial wealth w,*
- *$A_\xi(w)$ is the subset of random wealths that are almost surely anti-comonotonic with ξ_T (in other words which are almost surely a non-increasing function of ξ_T).*