Explicit Representation of Cost-Efficient Strategies

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- This talk is joint work with **Phelim Boyle** (Wilfrid Laurier University, Waterloo, Canada) and with **Steven Vanduffel** (Vrije Universiteit Brussel (VUB), Belgium).

- **Outline of the talk:**

  1. Characterization of optimal investment strategies for an investor with **law-invariant preferences** and a **fixed investment horizon**
  2. Optimal Design of Financial Products
  3. Extension to the case when investors have **state-dependent constraints**.

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Part I: Optimal portfolio selection for law-invariant investors

Characterization of optimal investment strategies for an investor with **law-invariant preferences** and a **fixed investment horizon**

- Optimal strategies are "cost-efficient".
- **Cost-efficiency** $\Leftrightarrow$ Minimum correlation with the state-price process $\Leftrightarrow$ Anti-monotonicity
- Explicit representations of the **cheapest** and **most expensive** strategies to achieve a given distribution.
- In the **Black-Scholes** setting,
  - *Optimality* of strategies increasing in S_T.
  - *Suboptimality* of path-dependent contracts.
  - How to *improve* structured products design.

Main Assumptions

- Consider an arbitrage-free market.
- Given a strategy with payoff X_T at time T. There exists Q, such that its price at 0 is

$$c(X_T) = \mathbb{E}_Q[e^{-rT} X_T]$$

- P ("physical measure") and Q ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT} \left(\frac{dQ}{dP}\right)_T, \quad \mathbf{c(X_T)} = \mathbb{E}_Q[e^{-rT} X_T] = \mathbb{E}_{\mathbf{P}}[\xi_{\mathbf{T}} \mathbf{X_T}].$$

We assume that all market participants agree on the state-price process ξ_T.

Cost-efficient strategies

A strategy (or a payoff) is cost-efficient

if any other strategy that generates the same distribution under P costs at least as much.

- Given a strategy with payoff X_T at time T and cdf F under the **physical measure** P.

The distributional price is defined as

$$PD(F) = \min_{\{Y \mid Y \sim F\}} \{\mathbb{E}[\xi_T Y]\} = \min_{\{Y \mid Y \sim F\}} c(Y)$$

- The strategy with payoff X_T is cost-efficient if

$$PD(F) = c(X_T)$$

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Literature

- Cox, J.C., Leland, H., 1982. "On Dynamic Investment Strategies," *Proceedings of the seminar on the Analysis of Security Prices*, **26**(2), U. of Chicago (published in 2000 in *JEDC*), **24**(11-12), 1859-1880.
- Dybvig, P., 1988a. "Distributional Analysis of Portfolio Choice," *Journal of Business*, **61**(3), 369-393.
- Dybvig, P., 1988b. "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market," *Review of Financial Studies*, **1**(1), 67-88.

Simple Illustration

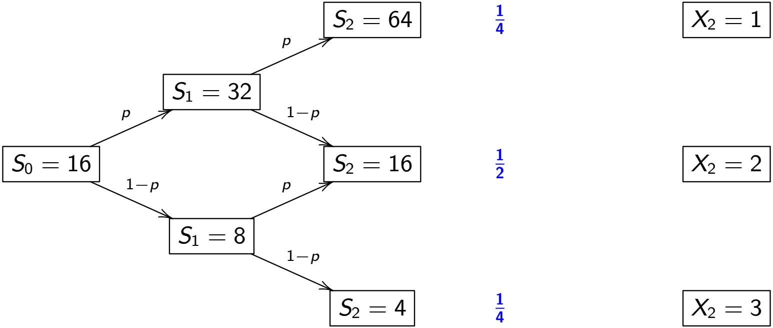
Example of

- $X_T \sim Y_T$ under P
- but with different costs

in a 2-period binomial tree. ($T = 2$)

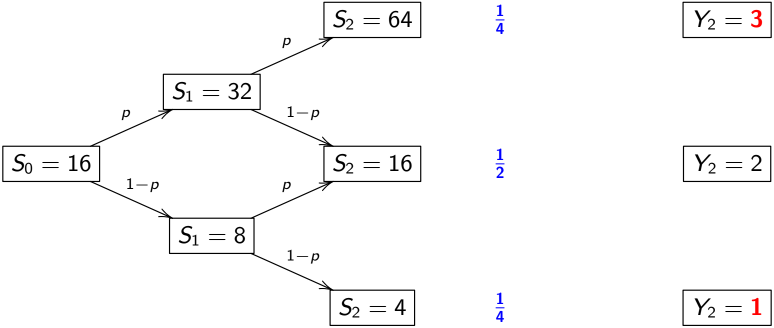
A simple illustration for X_2, a payoff at $T = 2$

Real-world probabilities: $p = \frac{1}{2}$



Y_2, a payoff at $T = 2$ distributed as X_2

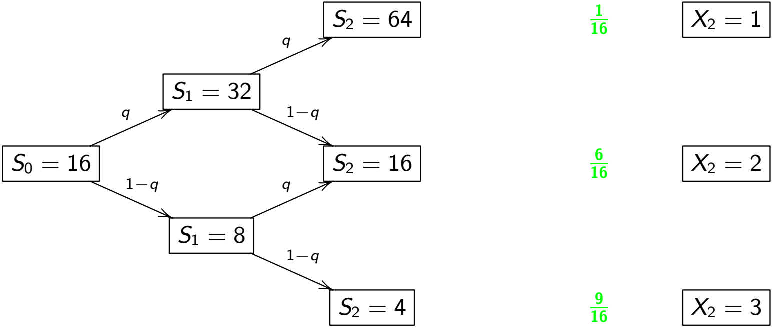
Real-world probabilities: $p = \frac{1}{2}$



X_2 and Y_2 have the same distribution under the physical measure

X_2, a payoff at $T = 2$

risk neutral probabilities: $q = \frac{1}{4}$.



$$c(X_2) \;=\; \text{Price of } X_2 = \left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2}$$

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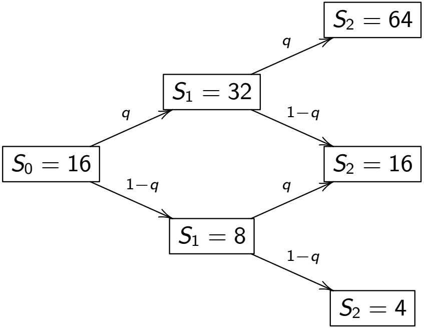

$\frac{1}{16}$ $\boxed{\mathbf{Y_2 = 3}}$

$\frac{6}{16}$ $\boxed{Y_2 = 2}$

$\frac{9}{16}$ $\boxed{\mathbf{Y_2 = 1}}$

$$c(Y_2) = \left(\frac{1}{16}3 + \frac{6}{16}2 + \frac{9}{16}1\right) = \frac{3}{2}$$

$$c(X_2) \ = \ \text{Price of } X_2 = \left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2}$$

Traditional Approach to Portfolio Selection

Consider an investor with **increasing law-invariant** preferences and a **fixed** horizon. Denote by X_T the investor's final wealth.

- Optimize a law-invariant objective function
  1. $\max_{\mathbf{X_T}} (\mathbf{E_P}[\mathbf{U}(\mathbf{X_T})])$ where U is increasing.
  2. Minimizing Value-at-Risk
  3. Probability target maximizing: $\max_{\mathbf{X_T}} \mathbf{P}(\mathbf{X_T} > \mathbf{K})$
  4. ...
- for a given **cost** (budget)

$$cost\ at\ 0 = E_Q[e^{-rT} X_T] = E_P[\xi_T X_T]$$

Find optimal strategy X_T^* $\Rightarrow$ Optimal cdf F of X_T^*

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Our Approach

Consider an investor with

- Law-invariant preferences
- Increasing preferences
- A fixed investment horizon

The optimal strategy must be **cost-efficient**.

Therefore $X_T^\star$ in the previous slide is cost-efficient.

Our approach: We characterize cost-efficient strategies

(This characterization can then be used to solve optimal portfolio problems by restricting the set of possible strategies).

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Sufficient Condition for Cost-efficiency

A subset A of $\mathbb{R}^2$ is anti-monotonic if

for any (x_1, y_1) and $(x_2, y_2) \in A$, $(x_1 - x_2)(y_1 - y_2) \leqslant 0$.

A random pair (X, Y) is anti-monotonic if

there exists an anti-monotonic set A of $\mathbb{R}^2$ such that $\mathbb{P}((X, Y) \in A) = 1$.

Theorem (Sufficient condition for cost-efficiency)

Any random payoff X_T with the property that (X_T, ξ_T) is **anti-monotonic** *is* **cost-efficient**.

Note the absence of additional assumptions on ξ_T (it holds in discrete and continuous markets) and on X_T (no assumption on non-negativity).

Explicit Representation for Cost-efficiency

Theorem

Consider the following optimization problem:

$$PD(F) = \min_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

Assume ξ_T is continuously distributed, *then the optimal strategy is*

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

Note that $X_T^\star \sim F$ and $X_T^\star$ is a.s. **unique** *such that*

$$PD(F) = c(X_T^\star) = \mathbb{E}[\xi_T X_T^\star]$$

Copulas and Sklar's theorem

The joint cdf of a couple (ξ_T, X) can be decomposed into 3 elements

- The marginal cdf of ξ_T: F_ξ
- The marginal cdf of X_T: F
- A copula C

such that

$$P(\xi_T \leqslant \xi, X_T \leqslant x) = C(F_\xi(\xi), F(x))$$

Idea of the proof (1/2)

Solving this problem amounts to finding bounds on copulas!

$$\min_{X_T} \mathbb{E}[\xi_T X_T]$$
$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \xi_T \sim F_\xi \end{cases}$$

The distribution F_ξ is known and depends on the financial market.
Let C denote a copula for (ξ_T, X).

$$\mathbb{E}[\xi_T X] = \int\int (1 - F_\xi(\xi) - F(x) + C(F_\xi(\xi), F(x)))dxd\xi, \quad (1)$$

Bounds for $\mathbb{E}[\xi_T X]$ are derived from bounds on C

$$\max(u + v - 1, 0) \leqslant C(u, v) \leqslant \min(u, v)$$

(Fréchet-Hoeffding Bounds for copulas)

Idea of the proof (2/2)

Consider a strategy with payoff X_T distributed as F. We define F^{-1} as follows:

$$F^{-1}(y) = \min\{x \ / \ F(x) \geqslant y\}.$$

ξ_T is continuously distributed. Let $U = F_\xi(\xi_T)$, then

$$E[F_\xi^{-1}(U)\, F_X^{-1}(1-U)] \leqslant E[F_\xi^{-1}(U)\, X] \leqslant E[F_\xi^{-1}(U)\, F_X^{-1}(U)]$$

In our setting, the cost of X_T is $c(X_T) = E[\xi_T X_T]$.

$$E[\xi_T F_X^{-1}(1-F_\xi(\xi_T))] \leqslant c(X_T) \leqslant E[\xi_T F_X^{-1}(F_\xi(\xi_T))]$$

Maximum price = Least efficient payoff

Theorem

Consider the following optimization problem:

$$\max_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

Assume ξ_T is continuously distributed. *The unique strategy $Z_T^\star$ that generates the same distribution as F with the highest cost can be described as follows:*

$$Z_T^\star = F^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

Path-dependent payoffs are inefficient

Corollary

To be cost-efficient, the payoff of the derivative has to be of the following form:

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

It becomes a European derivative written on S_T when the state-price process ξ_T can be expressed as a function of S_T. Thus path-dependent derivatives are in general not cost-efficient.

Corollary

Consider a derivative with a payoff X_T which could be written as

$$X_T = h(\xi_T)$$

Then X_T is cost efficient if and only if h is non-increasing.

Black-Scholes Model

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Then

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right) = a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = e^{\frac{\theta}{\sigma}(\mu-\frac{\sigma^2}{2})t-(r+\frac{\theta^2}{2})t}$ and $b = \frac{\mu-r}{\sigma^2}$.

Theorem

To be cost-efficient, the contract has to be a European derivative written on S_T and non-decreasing w.r.t. S_T (when $\mu > r$). In this case,

$$X_T^\star = F^{-1}\left(F_{S_T}\left(S_T\right)\right)$$

Geometric Asian contract in Black-Scholes model

Assume a strike K. The payoff of the Geometric Asian call is given by

$$X_T = \left(e^{\frac{1}{T}\int_0^T \ln(S_t)dt} - K\right)^+$$

which corresponds in the discrete case to $\left(\left(\prod_{k=1}^n S_{\frac{kT}{n}}\right)^{\frac{1}{n}} - K\right)^+$.

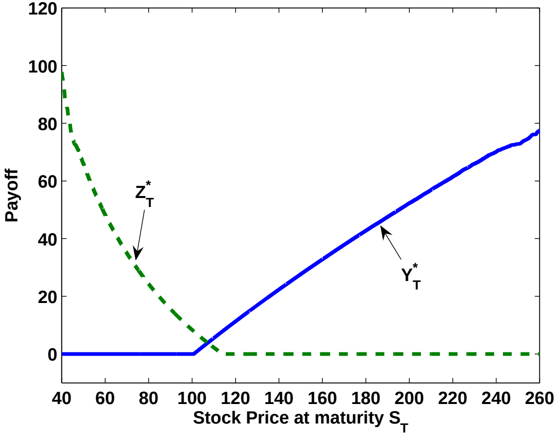
The efficient payoff that is distributed as the payoff X_T is a power call option

$$X_T^\star = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

where $d := S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$.

Similar result in the discrete case.

Example: Discrete Geometric Option



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100,\ n = 12.$

$$C(X_T^\star) = 5.77 < Price(geometric\ Asian) = 5.94 < C(Z_T^\star) = 9.03.$$

Put option in Black-Scholes model

Assume a strike K. The payoff of the put is given by

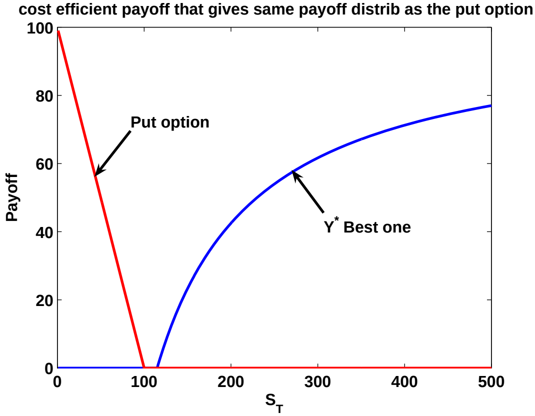
$$L_T = (K - S_T)^+ .$$

The payout that has the **lowest** cost and that has the same distribution as the put option payoff is given by

$$Y_T^\star = F_L^{-1}\left(F_{S_T}(S_T)\right) = \left(K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T}\right)^+ .$$

This type of power option “dominates” the put option.

Cost-efficient payoff of a put



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional price of the put $= 3.14$

Price of the put $= 5.57$

Efficiency loss for the put $=$ 5.57-3.14$=$ 2.43

Up and Out Call option in Black and Scholes model

Assume a strike K and a barrier threshold $H > K$. Its payoff is given by

$$L_T = (S_T - K)^+ \mathbb{1}_{\max_{0 \leqslant t \leqslant T}\{S_t\} \leqslant H}$$

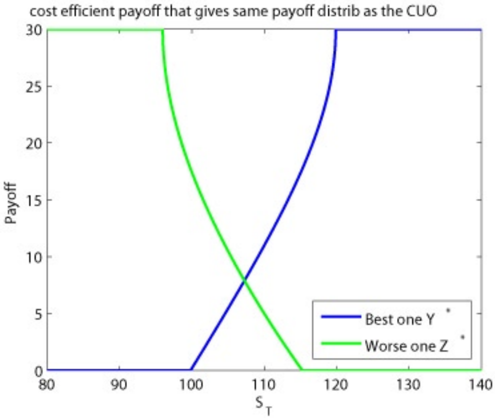
The payoff that has the **lowest** cost and is distributed such as the barrier up and out call option is given by

$$Y_T^\star = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

The payoff that has the **highest** cost and is distributed such as the barrier up and out call option is given by

$$Z_T^\star = F_L^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

Cost-efficient payoff of a Call up and out



With $\sigma = 20\%, \mu = 9\%, S_0 = 100,\ T = 1$ year, strike $K = 100,\ H = 130$

Distributional Price of the CUO $= 9.7374$

Price of CUO $= P_{cuo}$

Worse case $= 13.8204$

Efficiency loss for the CUO $= P_{cuo}$-9.7374

Link with First Stochastic Dominance

Theorem

Consider a payoff X_T with cdf F,

1. *Taking into account the initial cost of the derivative, the cost-efficient payoff $X_T^\star$ of the payoff X_T dominates X_T in the first order stochastic dominance sense :*

$$X_T - c(X_T)e^{rT} \prec_{fsd} X_T^\star - P_D(F)e^{rT}$$

2. *The dominance is strict unless X_T is a non-increasing function of ξ_T.*

Thus the result is true for any preferences that respect first stochastic dominance.

A Very Different Approach

Theorem

Any payoff X_T which cannot be expressed as a function of the state-price process ξ_T at time T is strictly dominated in the sense of second-order stochastic dominance by

$$H_T^{\star} = E\left[X_T \mid \sigma(\xi_T)\right] = g(\xi_T),$$

which is a function of ξ_T. Consequently in the Black and Scholes framework, any strictly path-dependent payoff is dominated by a path-independent payoff.

- Same cost.
- Different distribution.

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Example: the Lookback Option

Consider a lookback call option with strike K. The payoff on this option is given by

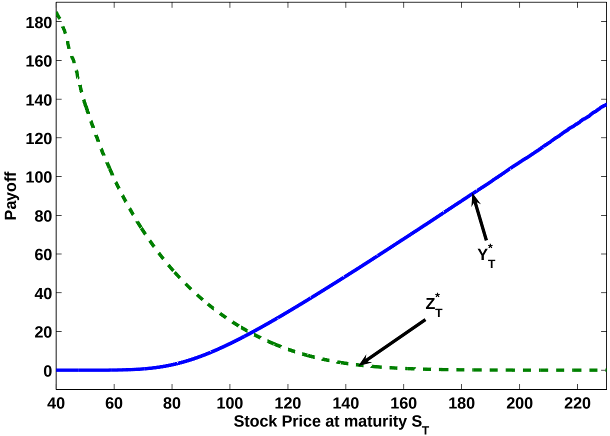
$$L_T = \left(\max_{0\leqslant t\leqslant T}\{S_t\} - K\right)^+.$$

The cost efficient payoff with the same distribution

$$Y_T^\star = F_L^{-1}\left(F_{S_T}\left(S_T\right)\right).$$

The payoff that has the highest cost and has the same distribution as the payoff L_T is given by $Z_T^\star = F_L^{-1}\left(1 - F_{S_T}\left(S_T\right)\right).$

Example: the Lookback Option



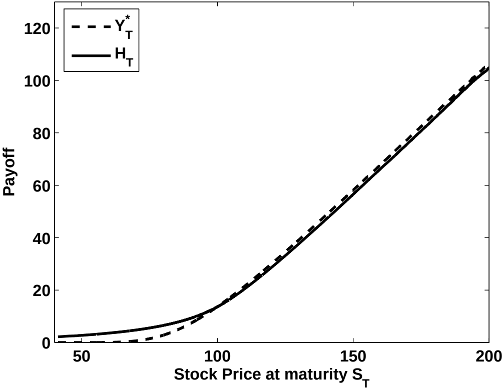
With

$\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional Price of the lookback $= 18.85$

Price of the lookback call $= 19.17$

Example: the Lookback Option



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Comparison of the two payoffs

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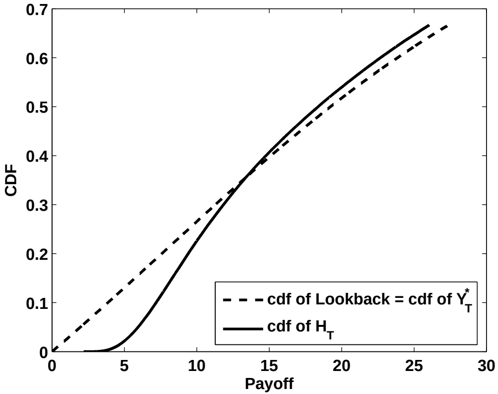


With $\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100$.

Comparison of the cdf of the two payoffs

Explaining the Demand for Inefficient Payoffs

1. **Other sources of uncertainty:** Stochastic interest rates or stochastic volatility
2. **Transaction costs, frictions**
3. **Intermediary consumption.**
4. Often we are looking at an **isolated contract**: the theory applies to the complete portfolio.
5. **State-dependent needs**
   - **Background risk:**
     - Hedging a long position in the market index S_T (background risk) by purchasing a put option,
     - the background risk can be path-dependent.
   - **Stochastic benchmark or other constraints:** If the investor wants to outperform a given (stochastic) benchmark Γ such that:

$$P\left\{\omega \in \Omega \,/\, W_T(\omega) > \Gamma(\omega)\right\} \geqslant \alpha.$$

Part 2:
Investment with State-Dependent Constraints

Problem considered so far

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\xi_T X_T\right].$$

A payoff that solves this problem is **cost-efficient**.

New Problem

$$\min_{\{Y_T \mid Y_T \sim F,\, \mathbb{S}\}} \mathbb{E}\left[\xi_T Y_T\right].$$

where $\mathbb{S}$ denotes a set of constraints. A payoff that solves this problem is called a $\mathbb{S}-$**constrained cost-efficient payoff**.

Copulas and Sklar's theorem

The joint cdf of a couple (S_T, X) can be decomposed into 3 elements

- The marginal cdf of S_T: G
- The marginal cdf of X_T: F
- A copula C

such that

$$P(S_T \leqslant s, X_T \leqslant x) = C(G(s), F(x))$$

How to formulate "state-dependent constraints"? (1/2)

Y_T and S_T have given distributions.

- The investor wants to ensure a **minimum** when the market falls

$$\mathbb{P}(Y_T > 100 \mid S_T < 95) = 0.8.$$

- This provides some additional information on the joint distribution between Y_T and S_T

$$\mathbb{P}(S_T < 95,\ Y_T > 100) = 0.2.$$

  $\Rightarrow$ information on the joint distribution of (ξ_T, Y_T) in the Black-Scholes framework.

- Note that $\mathbb{P}(\xi_T \leqslant x, Y_T \leqslant y) = \vartheta$, in other words

$$C(a, b) = \vartheta$$

  where $a = 1 - F_{S_T}(95), b = F_{S_T}(100)$ and $\vartheta = 0.2 - F_{S_T}(95) + F_{S_T}(100)$.

How to formulate "state-dependent constraints"? (2/2)

Y_T and S_T have given distributions.

- Y_T is **decreasing** in S_T when the stock S_T falls below some level (to justify the demand of a put option).
- Y_T is **independent** of S_T when S_T falls below some level.

All these constraints impose the strategy Y_T to pay out in given states of the world.

Formally

Goal: Find the **cheapest** possible payoff Y_T with the distribution F and which **satisfies additional constraints** of the form

$$\mathbb{P}(\xi_T \leqslant x, Y_T \leqslant y) = Q(F_{\xi_T}(x), F(y)),$$

with $x > 0, y \in \mathbb{R}$ and Q a given feasible function (for example a copula).

Each constraint gives information on the dependence between the state-price ξ_T and Y_T and is, for a given function Q, determined by the pair $(F_{\xi_T}(x), F(y))$.

Denote the finite or infinite set of all such constraints by $\mathbb{S}$.

Sufficient condition for the existence

Theorem

Let $t \in (0, T)$. If there exists a copula L satisfying $\mathbb{S}$ such that $L \leqslant C$ (pointwise) for all other copulas C satisfying $\mathbb{S}$ then the payoff $Y_T^\star$ given by

$$Y_T^\star = F^{-1}(f(\xi_T, \xi_t))$$

is a $\mathbb{S}$-constrained cost-efficient payoff. Here $f(\xi_T, \xi_t)$ is given by

$$f(\xi_T, \xi_t) = \left(\ell_{F_{\xi_T}(\xi_T)}\right)^{-1} \left[j_{F_{\xi_T}(\xi_T)}(F_{\xi_t}(\xi_t))\right],$$

where the functions $j_u(v)$ and $\ell_u(v)$ are defined as the first partial derivative for $(u, v) \to J(u, v)$ and $(u, v) \to L(u, v)$ respectively and where J denotes the copula for the random pair (ξ_T, ξ_t).

If (U, V) has a copula L then $\ell_u(v) = \mathbb{P}(V \leqslant v | U = u)$.

Example 1: $\mathbb{S} = \emptyset$ (no constraints)

From the Fréchet-Hoeffding bounds on copulas one has

$$\forall (u,v) \in [0,1]^2, \quad C(u,v) \geqslant \max\left(0,\ u+v-1\right).$$

Note that $L(u,v) := \max(0,\ u+v-1)$ is a copula.
Then one obtains $\ell_u(v) = 1$ if $v > 1-u$ and that $\ell_u(v) = 0$ if $v < 1-u$. Hence we find that $\ell_u^{-1}(p) = 1-u$ for all $0 < p \leqslant 1$ which implies that

$$f(\xi_t, \xi_T) = 1 - F_\xi(\xi_T).$$

It follows that $Y_T^\star$ is given by

$$Y_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

Existence of the optimum $\Leftrightarrow$ Existence of minimum copula

Theorem (Sufficient condition for existence of a minimal copula L)

Let $\mathbb{S}$ be a **rectangle** $[u_1, u_2] \times [v_1, v_2] \subseteq [0,1]^2$. *Then a minimal copula $L(u, v)$ satisfying $\mathbb{S}$ exists and is given by*

$$L(u, v) = \max\{0,\ u + v - 1,\ K(u, v)\}.$$

where $K(u, v) = \max_{(a,b)\in\mathbb{S}}\{Q(a, b) - (a - u)^+ - (b - v)^+\}$.

Proof in a note written with Xiao Jiang and Steven Vanduffel extending Tankov's result (JAP 2012).

Consequently the existence of a $\mathbb{S}-$constrained cost-efficient payoff is guaranteed when $\mathbb{S}$ is a rectangle. More generally it also holds when $\mathbb{S} \subseteq [0,1]^2$ satisfies a "monotonicity property" of the upper and lower "boundaries" and

$$\forall\,(u, v_0), (u, v_1) \in \mathcal{S},\ \left(u, \frac{v_0 + v_1}{2}\right) \in \mathcal{S}. \tag{2}$$

Theorem (Case of one constraint)

Assume that there is only one constraint (a, b) *in* $\mathbb{S}$ *and let* $\vartheta := Q(a, b)$, *Then the minimum copula L is*

$$L(u, v) = \max\left\{0,\ u + v - 1,\ \vartheta - (a - u)^{+} - (b - v)^{+}\right\}.$$

The $\mathbb{S}-$*constrained cost-efficient payoff* $Y_T^{\star}$ *exists and is unique. It can be expressed as*

$$Y_T^{\star} = F^{-1}\left(G(F_{\xi_T}(\xi_T))\right), \tag{3}$$

where $G : [0, 1] \to [0, 1]$ *is defined as* $G(u) = \ell_u^{-1}(1)$ *and can be written as*

$$G(u) = \begin{cases} 1 - u & \text{if } 0 \leqslant u \leqslant a - \vartheta, \\ a + b - \vartheta - u & \text{if } a - \vartheta < u \leqslant a, \\ 1 + \vartheta - u & \text{if } a < u \leqslant 1 + \vartheta - b, \\ 1 - u & \text{if } 1 + \vartheta - b < u \leqslant 1. \end{cases} \tag{4}$$

Example 2: $\mathbb{S}$ contains 1 constraint

Assume a Black-Scholes market. We suppose that the investor is looking for the payoff Y_T such that $Y_T \sim F$ (where F is the cdf of S_T) and satisfies the following constraint

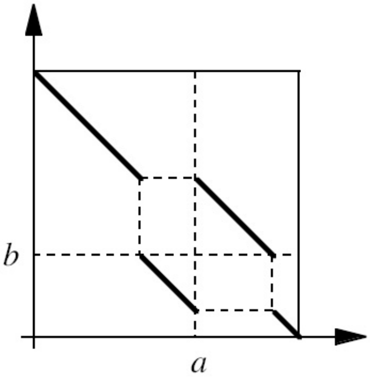
$$\mathbb{P}(S_T < 95,\ Y_T > 100) = 0.2.$$

The optimal strategy, where $a = 1 - F_{S_T}(95)$, $b = F_{S_T}(100)$ and $\vartheta = 0.2 - F_{S_T}(95) + F_{S_T}(100)$ is given by the previous theorem.

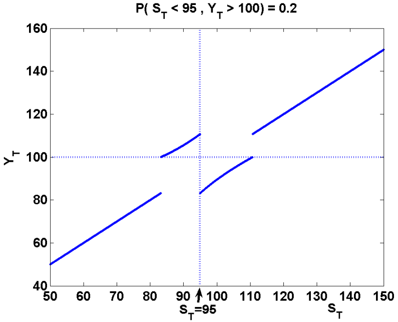
Its price is 100.2

Example 2: Illustration

Minimum Copula



Optimal Strategy



Example 3: $\mathbb{S}$ is infinite

A cost-efficient strategy with the same distribution F as S_T but such that it is decreasing in S_T when $S_T \leqslant \ell$ is unique a.s. Its payoff is equal to

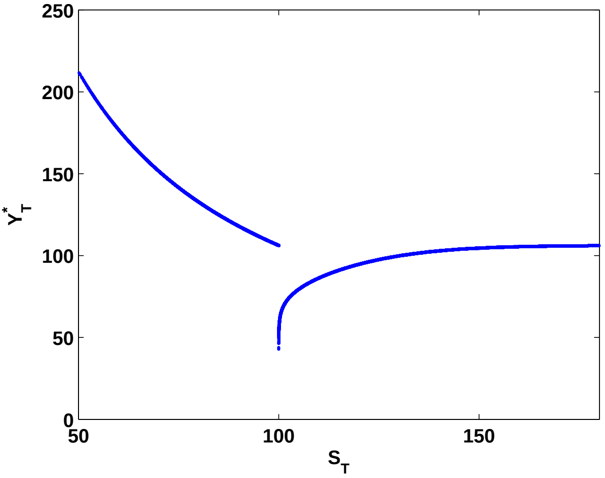
$$Y_T^\star = F^{-1}\left[G(F\left(S_T\right))\right],$$

where $G : [0,1] \to [0,1]$ is given by

$$G(u) = \begin{cases} 1-u & \text{if } 0 \leqslant u \leqslant F(\ell), \\ u - F(\ell) & \text{if } F(\ell) < u \leqslant 1. \end{cases}$$

The **constrained cost-efficient payoff** can be written as

$$Y_T^\star := F^{-1}\left[(1-F(S_T))\mathbb{1}_{S_T<\ell} + \left(F(S_T) - F(\ell)\right)\mathbb{1}_{S_T\geqslant\ell}\right].$$



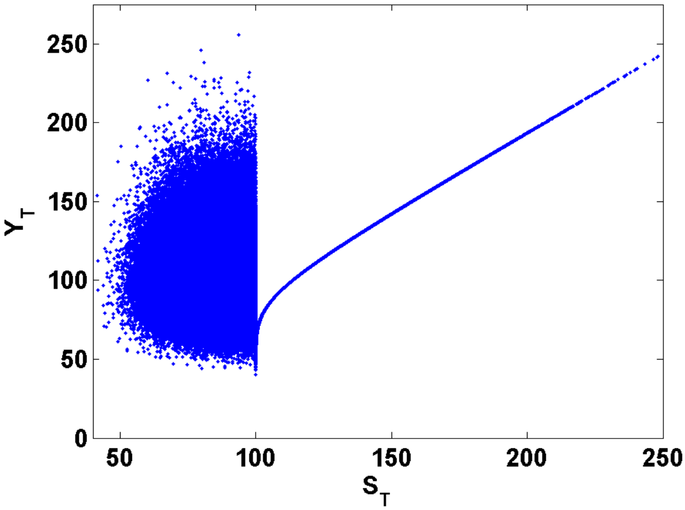
$Y_T^\star$ as a function of S_T. Parameters: $\ell = 100$, $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$ and $r = 0.03$. The price is 103.4.

Example 4: $\mathbb{S}$ is infinite

A cost-efficient strategy with the same distribution F as S_T but such that it is independent of S_T when $S_T \leqslant \ell$ can be constructed as

$$Y_T^\star = F^{-1}\left(\Phi\left(k(S_t, S_T)\right)\mathbb{1}_{S_T<\ell} + \left(\frac{F(S_T)-F(\ell)}{1-F(\ell)}\right)\mathbb{1}_{S_T\geqslant\ell}\right),$$

where $k(S_t, S_T) = \frac{\ln\left(\frac{S_t}{S_T^{t/T}}\right) - \left(1-\frac{t}{T}\right)\ln(S_0)}{\sigma\sqrt{t-\frac{t^2}{T}}}$ and $t \in (0, T)$ can be chosen freely (**No uniqueness and path-independence anymore**).



10,000 realizations of $Y_T^\star$ as a function of S_T where $\ell = 100$, $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$, $r = 0.03$ and $t = T/2$. Its price is 101.1

Conclusion

- Characterization of cost-efficient strategies.
- For a given investment strategy, we derive an explicit analytical expression for the cheapest and the most expensive strategies that have the same payoff distribution.
- Optimal investment choice under state-dependent constraints.
- **How to improve the design of structured products?**
  Simple contracts are usually **better**!!!
  In the presence of state-dependent constraints, optimal strategies
  - **are not always non-decreasing with the stock price S_T.**
  - **are not anymore unique and could be path-dependent.**

Further Research Directions / Work in Progress

- Using cost-efficiency to derive **bounds for insurance prices** derived from indifference utility pricing ("Bounds for Insurance Prices" with Steven Vanduffel)
- Extension to the presence of **stochastic interest rates** and application to executive compensation (work in progress with Jit Seng Chen and Phelim Boyle).
- Further extend the work on state-dependent constraints:
  1. Solve with **expectations constraints** between ξ_T and X_T.
  $$\mathbb{E}[g_i(\xi_T, X_T)] \in I_i$$
  where I_i is an interval, possibly reduced to a single value.
  2. Solve with the probability constraint of outperforming a benchmark
  $$\mathbb{P}(X_T > h(S_T)) \geqslant \varepsilon$$
  3. Extend the literature on optimal portfolio selection in specific models under state-dependent constraints.

Do not hesitate to contact me to get updated working papers!

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