# Explicit Representation of Cost Efficient Strategies

*Suboptimality of Path-dependent Strategies*

Carole Bernard (University of Waterloo)
& Phelim Boyle (Wilfrid Laurier University)

# Motivation / Context

- Starting point: work on popular US retail investment products. How to explain the demand for complex path-dependent contracts?
- Met with Phil Dybvig at the NFA in Sept. 2008.
- Path-dependent contracts are not "efficient" (JoB 1988, "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market" in RFS 1988).

# Motivation / Context

- Starting point: work on popular US retail investment products. How to explain the demand for complex path-dependent contracts?
- Met with Phil Dybvig at the NFA in Sept. 2008.
- Path-dependent contracts are not "efficient" (JoB 1988, "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market" in RFS 1988).

# Motivation / Context

- Starting point: work on popular US retail investment products. How to explain the demand for complex path-dependent contracts?
- Met with Phil Dybvig at the NFA in Sept. 2008.
- Path-dependent contracts are not "efficient" (JoB 1988, "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market" in RFS 1988).

## Some Assumptions

- Consider an arbitrage-free and complete market.
- Given a strategy with payoff $X_T$ at time $T$. There exists $Q$, such that its price at 0 is

$$P_X = E_Q[e^{-rT} X_T]$$

- $P$ ("physical measure") and $Q$ ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT} \left(\frac{dQ}{dP}\right)_T, \quad P_X = E_Q[e^{-rT} X_T] = E_P[\xi_T X_T].$$

## Motivation: Traditional Approach to Portfolio Selection

Investors have a strategy that will give them a final wealth $X_T$.
This strategy depends on the financial market and is random.

- They want to maximize the **expected utility** of their final wealth $X_T$
$$\max_{X_T}\left(E_P[U(X_T)]\right)$$
$U$: utility (increasing because individuals prefer more to less).
- They want to minimize the **cost of the strategy**
$$cost\ at\ 0 = E_Q[e^{-rT}X_T] = E_P[\xi_T X_T]$$

**Find optimal payoff $X_T$ $\Rightarrow$ Optimal cdf $F$ of $X_T$**

## Motivation: Traditional Approach to Portfolio Selection

Investors have a strategy that will give them a final wealth $X_T$.
This strategy depends on the financial market and is random.

- They want to maximize the **expected utility** of their final wealth $X_T$

$$\max_{X_T}\left(E_P[U(X_T)]\right)$$

  $U$: utility (increasing because individuals prefer more to less).

- They want to minimize the **cost of the strategy**

$$cost\ at\ 0 = E_Q[e^{-rT}X_T] = E_P[\xi_T X_T]$$

**Find optimal payoff $X_T$ $\Rightarrow$ Optimal cdf $F$ of $X_T$**

## Our Approach

- Given the cdf $F$ that the investor would like for his final wealth
- We give an explicit representation of the payoff $X_T$ such that
  - $X_T \sim F$ in the real world
  - $X_T$ corresponds to the cheapest strategy

# Outline of the presentation

- What is cost-efficiency?
- Path-dependent strategies/payoffs are not cost-efficient.
- **Explicit** construction of **efficient** strategies.
- Investors (with a fixed horizon and law-invariant preferences) should prefer to invest in path-independent payoffs:
  path-dependent exotic derivatives are usually not optimal!
- Examples: the put option and the geometric Asian option.

## Efficiency Cost

Dybvig (RFS 1988) explains how to compare two strategies by analyzing their respective efficiency cost.

**What is the "efficiency cost"?**

It is a criteria for evaluating payoffs independent of the agents' preferences.

# Efficiency Cost

- Given a strategy with payoff $X_T$ at time $T$, and initial price at time 0
$$P_X = E_P\left[\xi_T X_T\right]$$
- $F$ : $X_T$'s distribution under the **physical measure** $P$.

**The distributional price is defined as**

$$PD(F) = \min_{\{Y_T \mid Y_T \sim F\}} \left\{E_P\left[\xi_T Y_T\right]\right\}$$

**The "loss of efficiency" or "efficiency cost" is equal to:**

$$P_X - PD(F)$$

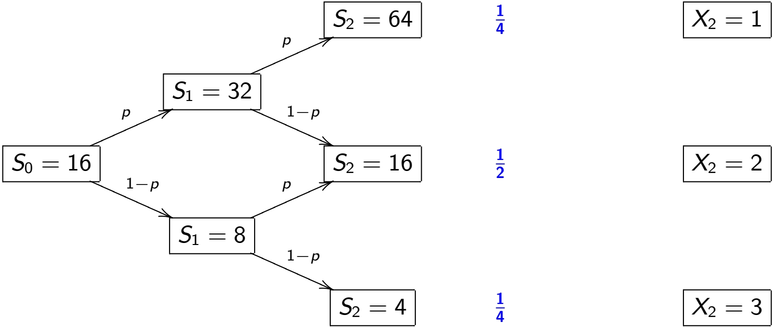
## A Simple Illustration

Let's illustrate what the "efficiency cost" is with a simple example. Consider :

- A market with 2 assets: a bond and a stock $S$.
- A discrete 2-period binomial model for the stock $S$.
- A strategy with payoff $X_T$ at the end of the two periods.
- An expected utility maximizer with utility function $U$.

## A simple illustration for $X_2$, a payoff at $T=2$

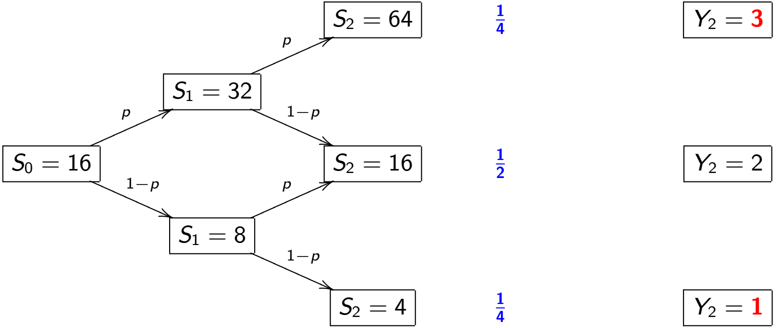
**Real-world** probabilities$=p=\frac{1}{2}$




$$E[U(X_2)] = \frac{U(1)+U(3)}{4} + \frac{U(2)}{2}$$

## $Y_2$, a payoff at $T = 2$ distributed as $X_2$

**Real-world** probabilities$=p = \frac{1}{2}$



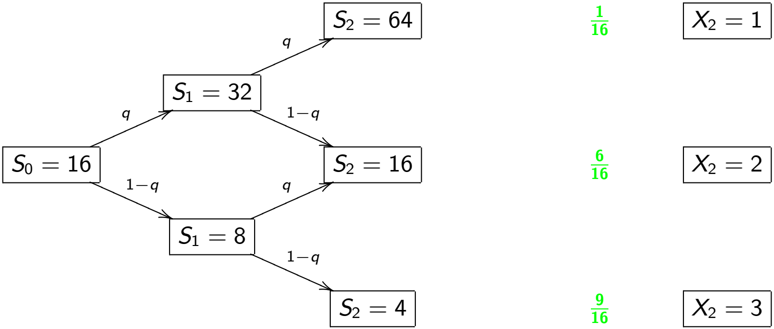

$$E[U(Y_2)] = \frac{U(3) + U(1)}{4} + \frac{U(2)}{2}$$

($X$ and $Y$ have the same distribution under the physical measure and thus the same utility)

# $X_2$, a payoff at $T=2$

**risk neutral**

probabilities$=q=\frac{1}{4}$.

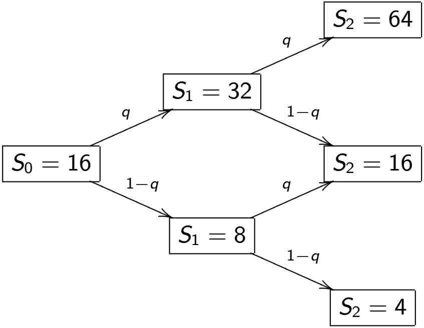



$$P_{X_2} = \text{Price of } X_2 = \left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2}$$

# $Y_2$, a payoff at $T = 2$

**risk neutral**

probabilities$=q=\frac{1}{4}$.




| | |
|---|---|
| $\frac{1}{16}$ | $\mathbf{Y_2 = 3}$ |
| $\frac{6}{16}$ | $Y_2 = 2$ |
| $\frac{9}{16}$ | $\mathbf{Y_2 = 1}$ |

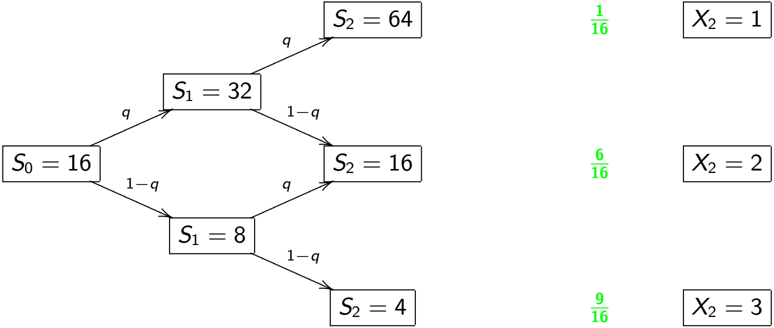
$$P_{Y_2} = \left(\frac{1}{16}3 + \frac{6}{16}2 + \frac{9}{16}1\right) = \frac{3}{2}$$

$$P_{X_2} \;=\; \text{Price of } X_2 = \left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2} \quad ,$$

## A simple illustration for $X_2$, a payoff at $T = 2$

**risk neutral**

probabilities$=q = \frac{1}{4}$.



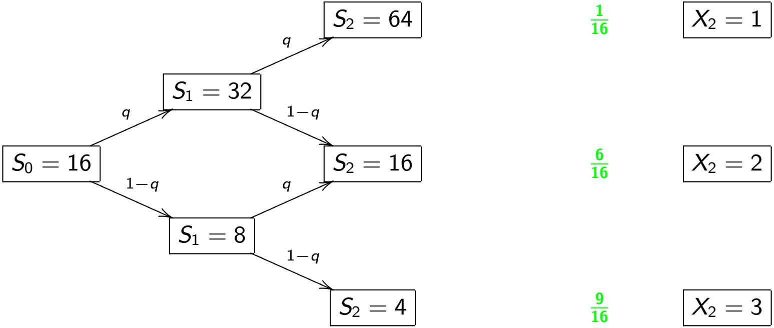

$$P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} \;=\; \text{Price of } X_2 = \frac{5}{2} \quad ,$$

## A simple illustration for $X_2$, a payoff at $T = 2$

**risk neutral**

probabilities$=q = \frac{1}{4}$.

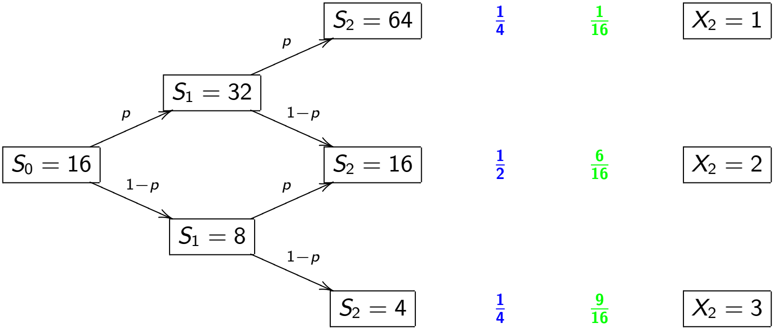



$$P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} = \text{Price of } X_2 = \frac{5}{2} \quad , \quad \text{Efficiency cost} = P_{X_2} - P_D$$

# A simple illustration for $X_2$, a payoff at $T = 2$

**Real-world** probabilities$=p = \frac{1}{2}$ and **risk neutral** probabilities$=q = \frac{1}{4}$.




$$E[U(X_2)] = \frac{U(1) + U(3)}{4} + \frac{U(2)}{2} \quad , \quad P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} = \text{Price of } X_2 = \frac{5}{2} \quad , \quad \text{Efficiency cost} = P_{X_2} - P_D$$

## Cost-efficiency

- The **cost of a strategy** (or of a financial investment contract) with terminal payoff $X_T$ is given by:

$$c(X_T) = E[\xi_T X_T]$$

- The **"distributional price"** of a cdf $F$ is defined as

$$PD(F) = \min_{\{Y \mid Y \sim F\}} \{c(Y)\}$$

where $\{Y \mid Y \sim F\}$ is the set of r.v. distributed as $X_T$ is.

**We want to find the strategy that realizes this minimum.**

## Minimum Cost-efficiency

Given a payoff $X_T$ with cdf $F$. We define its inverse $F^{-1}$ as follows:

$$F^{-1}(y) = \min\{x \;/\; F(x) \geq y\}.$$

**Theorem**

*Define*

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

*then $X_T^\star \sim F$ and $X_T^\star$ is a.s. unique such that*

$$PD(F) = c(X_T^\star)$$

Consider a strategy with payoff $X_T$ distributed as $F$. The cost of this strategy satisfies:

$$P_D(F) \leqslant c(X_T) \leqslant E[\xi_T F^{-1}(F_\xi(\xi_T))] = \int_0^1 F_\xi^{-1}(v)F^{-1}(v)dv$$

## The least efficient payoff

### Theorem

*Let $F$ be a cdf such that $F(0) = 0$. Consider the following optimization problem:*

$$\max_{\{Z \mid Z \sim F\}} \{c(Z)\}$$

*The strategy $Z_T^\star$ that generates the same distribution as $F$ with the highest cost can be described as follows:*

$$Z_T^\star = F^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

One needs

$$E[\xi_T F_X^{-1}(1 - F_\xi(\xi_T))] \leqslant E[\xi_T X_T] \leqslant E[\xi_T F_X^{-1}(F_\xi(\xi_T))]$$

It comes from the following property. Let $Z = F_Z^{-1}(U)$, then

$$E[F_Z^{-1}(U) F_X^{-1}(1 - U)] \leqslant E[F_Z^{-1}(U) X] \leqslant E[F_Z^{-1}(U) F_X^{-1}(U)]$$

## Path-dependent payoffs are inefficient

### Corollary

*To be cost-efficient, the payoff of the derivative has to be of the following form:*

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

*It becomes a European derivative written on $S_T$ as soon as the state-price process $\xi_T$ can be expressed as a function of $S_T$. Thus path-dependent derivatives are in general not cost-efficient.*

### Corollary

*Consider a derivative with a payoff $X_T$ which could be written as*

$$X_T = h(\xi_T)$$

*Then $X_T$ is cost efficient if and only if $h$ is non-increasing.*

## Black and Scholes Model

Under the physical measure $P$,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Under the risk neutral measure $Q$,

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t^Q$$

$S_t$ has a lognormal distribution.

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T = e^{-rT} a \left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = \exp\left(\frac{1}{2}Tb(r+\mu-\sigma^2) - rT\right)$ $b = \frac{\mu - r}{\sigma^2}$.

## Black and Scholes Model

**Any path-dependent financial derivative is inefficient**. Indeed

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T = e^{-rT}a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = \exp\left(\frac{1}{2}Tb(r+\mu-\sigma^2) - rT\right)$ $b = \frac{\mu-r}{\sigma^2}$.
**To be cost-efficient, the payoff has to be written as**

$$X^{\star} = F^{-1}\left(1 - F_{\xi}\left(a\left(\frac{S_T}{S_0}\right)^{-b}\right)\right)$$

It is a European derivative written on the stock $S_T$ (**and the payoff is increasing with** $S_T$ when $\mu > r$).

## Put option in Black and Scholes model

Assume a strike $K$. The payoff of the put is given by

$$L_T = (K - S_T)^+ .$$

The payoff that has the **lowest** cost and is distributed such as the put option is given by

$$Y_T^\star = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

## Put option in Black and Scholes model

Assume a strike $K$. The payoff of the put is given by

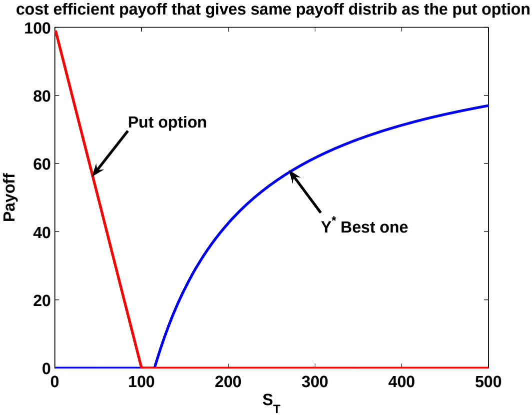
$$L_T = (K - S_T)^+ .$$

The cost-efficient payoff that will give the same distribution as a put option is

$$Y_T^\star = \left(K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T}\right)^+ .$$

This type of power option "dominates" the put option.

# Cost-efficient payoff of a put




With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional price of the put = 3.14

Price of the put = 5.57

Efficiency loss for the put = 5.57-3.14= 2.43

## Geometric Asian contract in Black and Scholes model

Assume a strike $K$. The payoff of the Gemoetric Asian call is given by

$$G_T = \left(e^{\frac{1}{T}\int_0^T \ln(S_t)dt} - K\right)^+$$

which corresponds in the discrete case to $\left(\left(\prod_{k=1}^n S_{\frac{kT}{n}}\right)^{\frac{1}{n}} - K\right)^+$.

The efficient payoff that is distributed as the payoff $G_T$ is given by

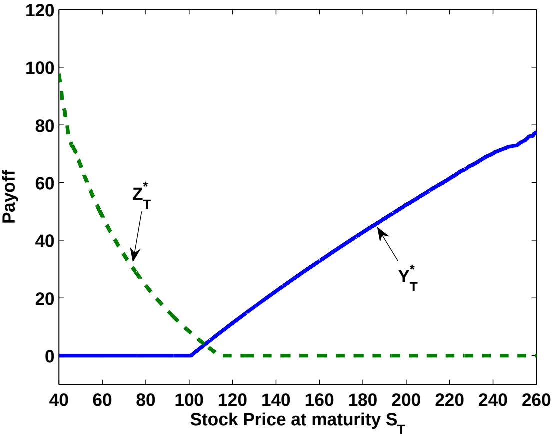
$$G_T^\star = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

where $d := S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$.

This payoff $G_T^\star$ is a power call option. If $\sigma = 20\%, \mu = 9\%$, $r = 5\%$, $S_0 = 100$. The price of this geometric Asian option is 5.94. The payoff $G_T^\star$ costs only 5.77.

Similar result in the discrete case.

# Example: the discrete Geometric option




With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100,\ n = 12$.

Price of the geometric Asian option = 5.94. The distributional price is 5.77.

The least-efficient payoff $Z_T^{\star}$ costs 9.03.

## Utility Independent Criteria

Denote by

- $X_T$ the final wealth of the investor,
- $V(X_T)$ the objective function of the agent,

Assumptions

1. **Agents' preferences depend only on the probability distribution of terminal wealth:** "law-invariant" preferences. (if $X_T \sim Z_T$ then: $V(X_T) = V(Z_T)$.)
2. **Agents prefer "more to less"**: if $c$ is a non-negative random variable $V(X_T + c) \geqslant V(X_T)$.
3. The market is perfectly liquid, no taxes, no transaction costs, no trading constraints (in particular short-selling is allowed).
4. The market is **arbitrage-free** and **complete**.

**For any inefficient payoff, there exists another strategy that these agents will prefer.**

# Utility Independent Criteria

Denote by

- $X_T$ the final wealth of the investor,
- $V(X_T)$ the objective function of the agent,

Assumptions

1. **Agents' preferences depend only on the probability distribution of terminal wealth:** "law-invariant" preferences. (if $X_T \sim Z_T$ then: $V(X_T) = V(Z_T)$.)
2. **Agents prefer "more to less"**: if $c$ is a non-negative random variable $V(X_T + c) \geqslant V(X_T)$.
3. The market is perfectly liquid, no taxes, no transaction costs, no trading constraints (in particular short-selling is allowed).
4. The market is **arbitrage-free** and **complete**.

**For any inefficient payoff, there exists another strategy that these agents will prefer.**

## Link with First Stochastic Dominance

### Theorem

*Consider a payoff $X_T$ with cdf $F$,*

1. *Taking into account the initial cost of the derivative, the cost-efficient payoff $X_T^\star$ of the payoff $X_T$ dominates $X_T$ in the first order stochastic dominance sense :*

$$X_T - c(X_T)e^{rT} \prec_{fsd} X_T^\star - P_D(F)e^{rT}$$

2. *The dominance is strict unless $X_T$ is a non-increasing function of $\xi_T$.*

Thus the result is true for any preferences that respect first stochastic dominance.

# A Very Different Approach

**Theorem**

*Any payoff $X_T$ which cannot be expressed as a function of the state-price process $\xi_T$ at time $T$ is strictly dominated in the sense of second-order stochastic dominance by*

$$H_T^\star = E\left[X_T \,|\, \sigma(\xi_T)\right] = g(\xi_T),$$

*which is a function of $\xi_T$. Consequently in the Black and Scholes framework, any strictly path-dependent payoff is dominated by a path-independent payoff.*

- Same cost.
- Different distribution.

# A Very Different Approach

**Theorem**

*Any payoff $X_T$ which cannot be expressed as a function of the state-price process $\xi_T$ at time $T$ is strictly dominated in the sense of second-order stochastic dominance by*

$$H_T^{\star} = E\left[X_T \,|\, \sigma(\xi_T)\right] = g(\xi_T),$$

*which is a function of $\xi_T$. Consequently in the Black and Scholes framework, any strictly path-dependent payoff is dominated by a path-independent payoff.*

- Same cost.
- Different distribution.

## Example: the Lookback Option

Consider a lookback call option with strike $K$. The payoff on this option is given by

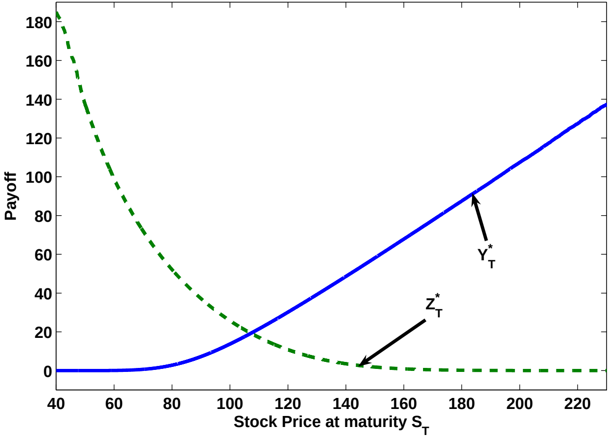
$$L_T = \left( \max_{0 \leqslant t \leqslant T} \{S_t\} - K \right)^+ .$$

The cost efficient payoff with the same distribution

$$Y_T^\star = F_L^{-1} \left(1 - F_\xi \left(\xi_T\right)\right).$$

The payoff that has the highest cost and has the same distribution as the payoff $L_T$ is given by $Z_T^\star = F_L^{-1} \left(F_\xi \left(\xi_T\right)\right).$

# Example: the Lookback Option



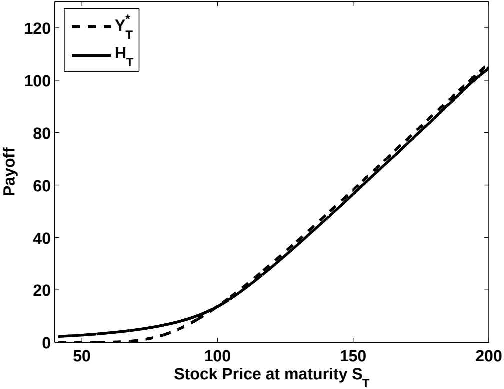

With

$\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100.$

Distributional Price of the lookback = 18.85

Price of the lookback call = 19.17

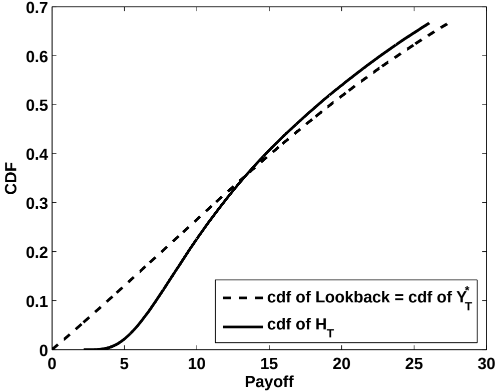
## Example: the Lookback Option




With $\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100$.

Comparison of the two payoffs

# Example: the Lookback Option




With $\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100$.

Comparison of the cdf of the two payoffs

# Explaining the Demand for Inefficient Payoffs

1. State-dependent needs
   - **Background risk:**
     - Hedging a long position in the market index $S_T$ (background risk) by purchasing a put option $P_T$,
     - the background risk can be path-dependent.
   - **Stochastic benchmark or other constraints:** If the investor wants to outperform a given (stochastic) benchmark $\Gamma$ such that:
   $$P\left\{\omega \in \Omega \,/\, W_T(\omega) > \Gamma(\omega)\right\} \geqslant \alpha.$$
   - **Intermediary consumption.**
2. **Other sources of uncertainty:** the state-price process is not always a monotonic function of $S_T$ (non-Markovian interest rates for instance)
3. **Transaction costs, frictions:** Preference for an available inefficient contract rather than a cost-efficient payoff that one needs to replicate.

## Conclusion

- A preference free framework for ranking different investment strategies.
- For a given investment strategy, we derive an explicit analytical expression
  1. for the cheapest strategy that has the same payoff distribution.
  2. for the most expensive strategy that has the same payoff distribution.
- There are strong connections between this approach and stochastic dominance rankings.

This may be useful for improving the design of financial products.

## References


- Bernard, C., Maj, M., and Vanduffel, S., 2010. "Improving the Design of Financial Products in a Multidimensional Black-Scholes Market," *NAAJ, forthcoming*.
- Cambanis S., Simons J. and Stout W. 1976. "Inequalities for $Ek(X, Y)$ when the marginals are fixed". Z. Wahrscheinlichkeitstheorie Verw. Geb., 36:285-294.
- Cox, J.C., Leland, H., 1982. "On Dynamic Investment Strategies," *Proceedings of the seminar on the Analysis of Security Prices*, **26**(2), University of Chicago. (published in 2000 in *Journal of Economic Dynamics and Control*, **24**(11-12), 1859-1880.
- Dybvig, P., 1988a. "Distributional Analysis of Portfolio Choice," *Journal of Business*, **61**(3), 369-393.
- Dybvig, P., 1988b. "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market," *RFS*.
- Goldstein, D.G., Johnson, E.J., Sharpe, W.F., 2008. "Choosing Outcomes versus Choosing Products: Consumer-focused Retirement Investment Advice," *Journal of Consumer Research*, **35**(3), 440-456.