# Prices and Asymptotics of Variance Swaps

**Carole Bernard**
**Zhenyu (Rocky) Cui**

University of Waterloo

WatRISQ
Waterloo Research Institute
in Insurance, Securities
& Quantitative Finance

**Beirut, May 2013.**

# Outline

- Motivation
- Convex order conjecture
- Discrete variance swaps: prices and asymptotics
- Conclusion & Future Directions

# Variance Swap

- A **variance swap** is an OTC contract:

$$\text{Notional } \times \left( \frac{1}{T} \text{ Realized Variance} - \text{Strike} \right)$$

- Realized Variance: **RV** $= \sum_{i=0}^{n-1} \left( \ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2$ with $0 = t_0 < t_1 < ... < t_n = T$.
- Quadratic Variation: **QV** $= \lim_{n \to \infty, \max_{i=0,1,\dots,n-1} (t_{i+1} - t_i) \to 0} RV$.
- In practice, variance swaps are **discretely** sampled but it is typically easier to compute the **continuously** sampled in popular stochastic volatility models.
- Question: Finding “fair” strikes so that the initial value of the contract is 0.

# Model Setting (1/2)

- Under the risk-neutral probability measure $Q$,

$$\textbf{(M)} \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)} \\ dV_t &= \mu(V_t)dt + \sigma(V_t)dW_t^{(2)} \end{cases}$$

where $\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$.

# Three Stochastic Volatility Models

Assume $\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$.

- The **correlated** Heston model:

$$(\mathbf{H}) \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)}, \\ dV_t &= \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}dW_t^{(2)} \end{cases}$$

- The **correlated** Hull-White model:

$$(\mathbf{HW}) \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)}, \\ dV_t &= \mu V_t dt + \sigma V_t dW_t^{(2)} \end{cases}$$

- The **correlated** Schöbel-Zhu model:

$$(\mathbf{SZ}) \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + V_t dW_t^{(1)} \\ dV_t &= \kappa(\theta - V_t)dt + \gamma dW_t^{(2)} \end{cases}$$

# Model Setting (2/2)

- Under the risk-neutral probability measure $Q$,

$$(\mathbf{M}) \quad \begin{cases} \frac{dS_t}{S_t} & = & rdt + \sqrt{V_t}dW_t^{(1)} \\ dV_t & = & \mu(V_t)dt + \sigma(V_t)dW_t^{(2)} \end{cases}$$

where $\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$.

- The fair strike of the **"discrete variance swap"** is

$$\mathbf{K_d^M(n)} := \frac{1}{T}\mathbb{E}\left[\sum_{i=0}^{n-1}\left(\ln\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = \frac{1}{T}\mathbb{E}[RV]$$

- The fair strike of the **"continuous variance swap"** is

$$\mathbf{K_c^M} := \frac{1}{T}\mathbb{E}\left[\int_0^T V_s ds\right] = \frac{1}{T}\mathbb{E}[QV]$$

# Contributions

- A general expression for the fair strike of a discrete variance swap in the time-homogeneous stochastic volatility model:
- Application in three popular stochastic volatility models
  (1) Heston model: **more explicit** than Broadie and Jain (2008).
  (2) Hull-White model: a **new** closed-form formula.
  (3) Schöbel-Zhu model: : a **new** closed-form formula.
- Asymptotic expansion of the fair strike with respect to $n$, $T$, vol of vol...
- A counter-example to the "**Convex Order Conjecture**".

# Convex Order Conjecture

- Notations:
  (1) $RV = \sum_{i=0}^{n-1} (\log(S_{t_{i+1}}/S_{t_i}))^2$: **discrete realized variance** for a partition of $[0, T]$ with $n+1$ points;

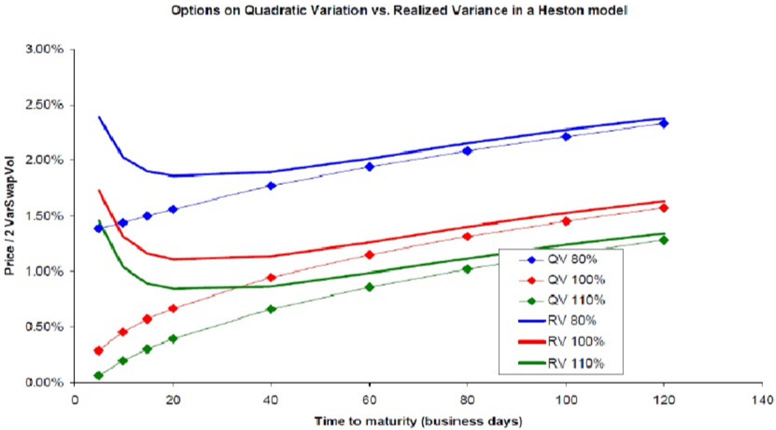
  (2) $QV = \int_0^T V_s ds$: **continuous quadratic variation**.

- Usual practice: **approximate** $\mathbb{E}[f(RV)]$ with $\mathbb{E}[f(QV)]$, see Jarrow et al (2012).

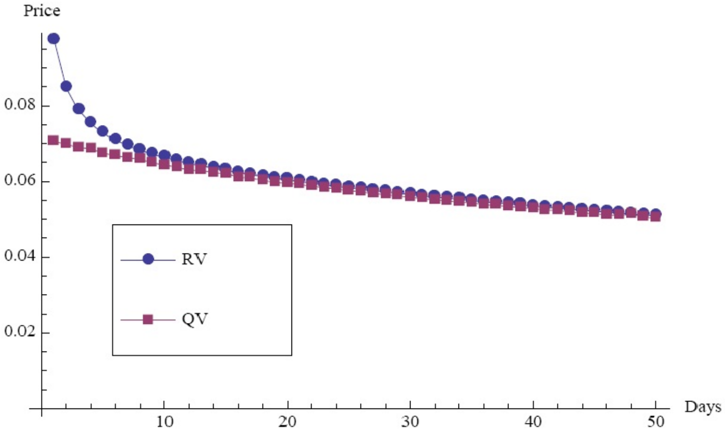
- Bülher (2006): "while the approximation of realized variance via quadratic variation works very well for variance swaps, it is not sufficient for **non-linear payoffs with short maturities**".

- Call option on RV: $(RV - K)^+$;

  Call option on QV: $(QV - K)^+$.




Plot from Bühler (2006b).




Plot from Keller-Ressel and Muhle-Karbe (2010). Shows ATM call prices in Kou's jump-diffusion model.

# Convex Order Conjecture (Cont'd)

- The convex-order conjecture (Keller-Ressel (2011)):

  "**The price of a call option on realized variance is higher than the price of a call option on quadratic variation**"

- Equivalently, $\mathbb{E}[f(RV)] \geqslant \mathbb{E}[f(QV)]$ where $f$ is convex.

- When $f(x) = x$, our closed-form expression shows that when the correlation between the underlying and its variance is positive, it is possible to observe $K_d^M(n) < K_c^M$ (Illustrated by examples in Heston, Hull-White and Schöbel-Zhu models $(M)$).

# Conditional Black-Scholes Representation

- Recall

$$\textbf{(M)} \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)} \\ dV_t &= \mu(V_t)dt + \sigma(V_t)dW_t^{(2)} \end{cases}$$

- Cholesky decomposition: $dW_t^{(1)} = \rho dW_t^{(2)} + \sqrt{1-\rho^2}dW_t^{(3)}$.
- Key representation of the log stock price

$$\ln(S_T) = \ln(S_0) + rT - \frac{1}{2}\int_0^T V_t dt$$
$$+\rho\left(f(V_T) - f(V_0) - \int_0^T h(V_t)dt\right) + \sqrt{1-\rho^2}\int_0^T \sqrt{V_t}dW_t^{(3)}$$

where $f(v) = \int_0^v \frac{\sqrt{z}}{\sigma(z)}dz$, $h(v) = \mu(v)f'(v) + \frac{1}{2}\sigma^2(v)f''(v)$.

## Proposition

*Under some technical conditions,* $(\Delta = \frac{T}{n})$:,

$$\mathbb{E}\left[\left(\ln \frac{S_{t+\Delta}}{S_t}\right)^2\right] = \mathbf{r}^2\Delta^2 - \mathbf{r}\Delta \int_t^{t+\Delta} \mathbb{E}\left[V_s\right] ds$$

$$+ \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta} V_s ds\right)^2\right] + (1-\rho^2)\int_t^{t+\Delta} \mathbb{E}\left[V_s\right] ds$$

$$+ \rho^2 \mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))^2\right] + \rho^2 \mathbb{E}\left[\left(\int_t^{t+\Delta} h(V_s) ds\right)^2\right]$$

$$+ \rho \mathbb{E}\left[\int_t^{t+\Delta} h(V_s) ds \int_t^{t+\Delta} V_s ds\right]$$

$$- \rho \mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t)) \int_t^{t+\Delta} (2\rho h(V_s) + V_s) ds\right].$$

# Sensitivity w.r.t. interest rate

## Proposition (Sensitivity to $r$)

*The fair strike of the discrete variance swap:*

$$K_d^M(n) = b^M(n) - \frac{T}{n} K_c^M r + \frac{T}{n} r^2,$$

*where $b^M(n)$ does* **not** *depend on $r$.*

$$\frac{dK_d^M(n)}{dr} = \frac{T}{n}(2r - K_c^M)$$

*$K_d^M(r)$ reaches* **minimum** *when $r^* = \frac{K_c^M}{2}$.*

# Three Stochastic Volatility Models

Assume $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$.

- The **correlated** Heston model:

$$\textbf{(H)} \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t} dW_t^{(1)}, \\ dV_t &= \kappa(\theta - V_t)dt + \gamma \sqrt{V_t} dW_t^{(2)} \end{cases}$$

- The **correlated** Hull-White model:

$$\textbf{(HW)} \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t} dW_t^{(1)}, \\ dV_t &= \mu V_t dt + \sigma V_t dW_t^{(2)} \end{cases}$$

- The **correlated** Schöbel-Zhu model:

$$\textbf{(SZ)} \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + V_t dW_t^{(1)} \\ dV_t &= \kappa(\theta - V_t)dt + \gamma dW_t^{(2)} \end{cases}$$

# Heston Model

The fair strike of the **discrete** variance swap is

$$\mathbf{K_d^H(n)} = \frac{1}{8n\kappa^3 T}\Big\{2\kappa T\left(\kappa^2 T(\theta-2r)^2 + n\theta\left(4\kappa^2 - 4\rho\kappa\gamma+\gamma^2\right)\right)$$
$$+n\left(\gamma^2(\theta-2V_0)+2\kappa(V_0-\theta)^2\right)\left(e^{-2\kappa T}-1\right)\frac{1-e^{\frac{\kappa T}{n}}}{1+e^{\frac{\kappa T}{n}}}$$
$$+4(V_0-\theta)\left(n\left(2\kappa^2+\gamma^2-2\rho\kappa\gamma\right)+\kappa^2 T(\theta-2r)\right)\left(1-e^{-\kappa T}\right)$$
$$-2n^2\theta\gamma(\gamma-4\rho\kappa)\left(1-e^{-\frac{\kappa T}{n}}\right)+4(V_0-\theta)\kappa T\gamma(\gamma-2\rho\kappa)\frac{1-e^{-\kappa T}}{1-e^{\frac{\kappa T}{n}}}\Big\}$$

The fair strike of the **continuous** variance swap is

$$\mathbf{K_c^H} = \frac{1}{T}\mathbb{E}\left[\int_0^T V_s ds\right] = \theta + (1-e^{-\kappa T})\frac{V_0-\theta}{\kappa T}.$$

# Hull-White Model

The fair strike of the **discrete** variance swap is

$$\mathbf{K_d^{HW}(n)} = \frac{r^2 T}{n} + \frac{V_0}{\mu T}\left(1 - \frac{rT}{n}\right)\left(e^{\mu T} - 1\right)$$
$$- \frac{V_0^2\left(e^{\left(2\mu+\sigma^2\right)T} - 1\right)\left(e^{\frac{\mu T}{n}} - 1\right)}{2T\mu(\mu+\sigma^2)\left(e^{\frac{\left(2\mu+\sigma^2\right)T}{n}} - 1\right)} + \frac{V_0^2\left(e^{\left(2\mu+\sigma^2\right)T} - 1\right)}{2T(2\mu+\sigma^2)(\mu+\sigma^2)}$$
$$+ \frac{8\rho\left(e^{\frac{3(4\mu+\sigma^2)T}{8}} - 1\right) V_0{}^{3/2}\sigma(e^{\frac{\mu T}{n}} - 1)}{\mu T\left(4\mu+3\sigma^2\right)\left(e^{\frac{3(4\mu+\sigma^2)T}{8n}} - 1\right)} - \frac{64\rho\left(e^{\frac{3(4\mu+\sigma^2)T}{8}} - 1\right) V_0{}^{3/2}\sigma}{3T(4\mu+\sigma^2)\left(4\mu+3\sigma^2\right)}$$

The fair strike of the **continuous** variance swap is

$$\mathbf{K_c^{HW}} = \frac{1}{T}\mathbb{E}\left[\int_0^T V_s ds\right] = \frac{V_0}{T\mu}(e^{\mu T} - 1).$$

# Schöbel-Zhu Model

The fair strike of the **discrete** variance swap is explicit but too complicated to appear on a slide.
The fair strike of the **continuous** variance swap is

$$\mathbf{K_c^{SZ}} = \frac{\gamma^2}{2\kappa} + \theta^2 + \left(\frac{(V_0-\theta)^2}{2\kappa T} - \frac{\gamma^2}{4\kappa^2 T}\right)(1-e^{-2\kappa T}) + \frac{2\theta(V_0-\theta)}{\kappa T}(1-e^{-\kappa T}).$$

# Heston model: Expansion w.r.t $n$

$$\mathbf{K_d^H(n)} = \mathbf{K_c^H} + \frac{\mathbf{a_1^H}}{\mathbf{n}} + \mathcal{O}\left(\frac{\mathbf{1}}{\mathbf{n^2}}\right).$$

where $\mathbf{a_1^H}$ is a **linear and decreasing** function of $\rho$:[1]

$$\mathbf{a_1^H} \geqslant \mathbf{0} \quad \Longleftrightarrow \quad \rho \leqslant \rho_{\mathbf{0}}^{\mathbf{H}}$$

where

$$\rho_0^H = \frac{r^2 T - r K_c^H T + \left(\frac{\theta^2}{4} + \frac{\theta\gamma^2}{8\kappa}\right) T + c_1}{-\left(\frac{\gamma(\theta - V_0)}{2\kappa}(1 - e^{-\kappa T}) - \frac{\theta \gamma T}{2}\right)}.$$

[1] Explicit expression of $a_1^H$ is in Proposition 5.1, Bernard and Cui (2012).

# Hull-White model: Expansion w.r.t $n$

$$\mathbf{K_d^{HW}(n)} = \mathbf{K_c^{HW}} + \frac{\mathbf{a_1^{HW}}}{\mathbf{n}} + \mathcal{O}\left(\frac{\mathbf{1}}{\mathbf{n^2}}\right)$$

where $\mathbf{a_1^{HW}}$ is a **linear and decreasing** function of $\rho$:[2]

$$\mathbf{a_1^{HW}} \geqslant \mathbf{0} \quad \Longleftrightarrow \quad \rho \leqslant \rho_{\mathbf{0}}^{\mathbf{HW}}$$

where

$$\rho_0^{HW} = \frac{3(4\mu+\sigma^2)\left(r^2T - rK_c^{HW}T + \frac{V_0^2}{4}\frac{e^{(2\mu+\sigma^2)T}-1}{2\mu+\sigma^2}\right)}{4\sigma V_0^{\frac{3}{2}}\left(e^{\frac{3}{8}(4\mu+\sigma^2)T}-1\right)} > 0.$$

[2] Explicit expression of $a_1^{HW}$ is in Proposition 5.3, Bernard and Cui (2012).

# Schöbel-Zhu model: Expansion w.r.t $n$

The asymptotic behavior of the fair strike of a discrete variance swap in the Schöbel-Zhu model is given by

$$\mathbf{K_d^{SZ}(n)} = \mathbf{K_c^{SZ}} + \frac{\mathbf{a_1^{SZ}}}{\mathbf{n}} + \mathcal{O}\left(\frac{\mathbf{1}}{\mathbf{n^2}}\right),$$

where

$$a_1^{SZ} = r^2 T - rTK_c^{SZ} + d_1 + d_2 \frac{\gamma}{2\kappa}\rho. \quad (1)$$

and where $d_1$ and $d_2$ are explicit.

# Other Expansions

Given that the expressions are explicit, it is straightforward to obtain expansions for the discrete variance swaps as a function of the different parameters, and for example with respect to the maturity or to the volatility of volatility.

# Expansion of the fair strike for small maturity $T$

In the **Heston model**, an expansion of $K_d^H(n)$ when $T \to 0$ is

$$\mathbf{K_d^H(n) = V_0 + b_1^H T + b_2^H T^2} + \mathcal{O}\left(\mathbf{T^3}\right)$$

where

$$b_1^H = \frac{\kappa(\theta - V_0)}{2} + \frac{1}{4n}\left((V_0 - 2r)^2 - 2\gamma V_0 \rho\right)$$

$$b_2^H = \frac{\kappa^2(V_0 - \theta)}{6} + \frac{(V_0 - \theta)\kappa(\gamma\rho + 2r - V_0) + \frac{\gamma^2 V_0}{2}}{4n} + \frac{\gamma\rho\kappa(V_0 + \theta) - \frac{\gamma^2 V_0}{2}}{12n^2}.$$

and we have

$$\mathbf{K_d^H(n) - K_c^H = \frac{1}{4n}}\left(\mathbf{(V_0 - 2r)^2 - 2\rho\gamma V_0}\right)\mathbf{T} + \mathcal{O}(\mathbf{T^2}).$$

# Expansion of the fair strike for small maturity

In the **Hull-White model**, an expansion of $K_d^{HW}(n)$ when $T \to 0$ is

$$\mathbf{K_d^{HW}(n) = V_0 + b_1^{HW} T + b_2^{HW} T^2} + \mathcal{O}\left(\mathbf{T^3}\right)$$

where

$$b_1^{HW} = \frac{V_0\,\mu}{2} + \frac{1}{4n}\left((V_0 - 2r)^2 - 2\rho {V_0}^{3/2}\sigma\right)$$

$$b_2^{HW} = \frac{V_0\mu^2}{6} + \frac{V_0}{4n}\left(\frac{\sigma^2 V_0}{2} - \frac{3\rho\, {V_0}^{1/2}\sigma(\sigma^2+4\mu)}{8} + \mu(V_0 - 2r)\right) + \frac{{V_0}^{3/2}\sigma\left(\rho(3\sigma^2 - 4\mu) - 4\sigma\sqrt{V_0}\right)}{96n^2}$$

Note also

$$\mathbf{K_d^{HW}(n) - K_c^{HW} = \frac{1}{4n}}\left(\mathbf{(V_0 - 2r)^2 - 2\rho {V_0}^{3/2}}\sigma\right)\mathbf{T} + \mathcal{O}(\mathbf{T^2}).$$

# Expansion of the fair strike for small maturity

In the **Schöbel-Zhu model**, an expansion of $K_d^{HW}(n)$ when $T \to 0$ is In the Schöbel-Zhu model, $K_d^{SZ}(n)$ can be expanded when $T \to 0$ as

$$\mathbf{K_d^{SZ}(n) = V_0^2 + b_1^{SZ} T + \mathcal{O}(T^2)} \tag{2}$$

where

$$b_1^{SZ} = \kappa V_0(\theta - V_0) + \frac{\gamma^2}{2} + \frac{1}{n}\left(r^2 - rV_0^2 + \frac{V_0^2(V_0^2 - 4\rho\gamma)}{4}\right)$$

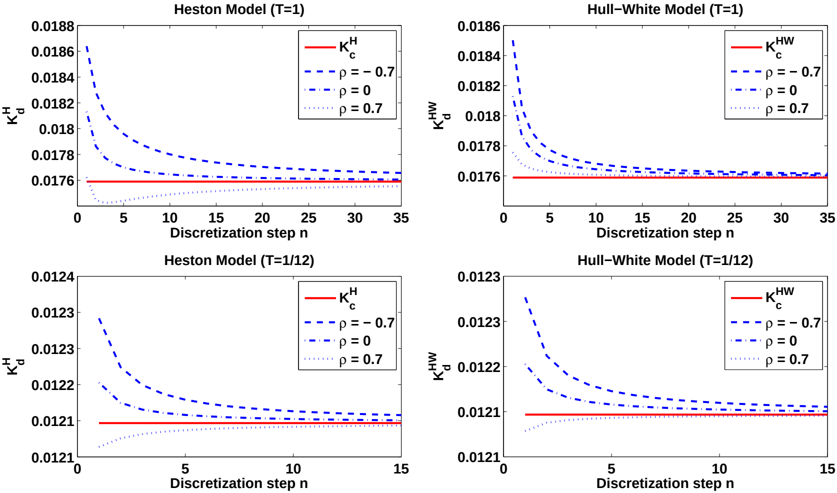
Note also

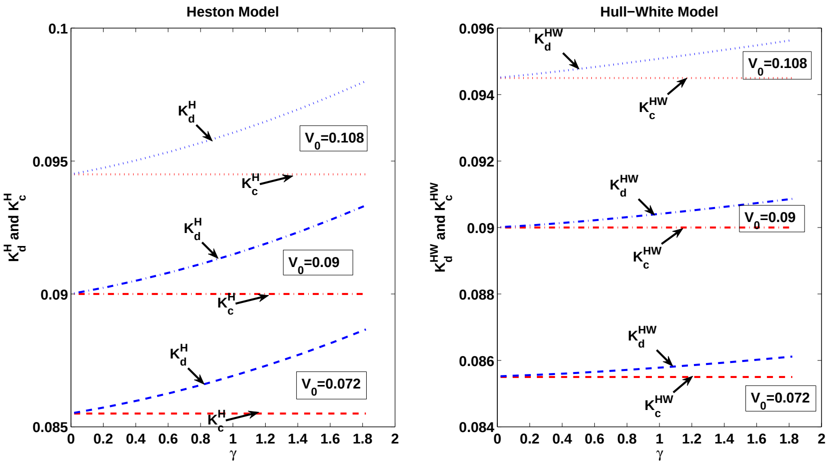
$$\mathbf{K_d^{SZ}(n) - K_c^{SZ} = \frac{1}{4n}\left((V_0^2 - 2r)^2 - 4\rho V_0^2\gamma\right) T + \mathcal{O}(T^2).}$$

# Parameters

- **Heston model:** First set of parameters from Broadie and Jain (2008). Second set is when $T = 1/12$.
- **Hull-White model:** obtain $\mu$ by numerically solving $K_c^H = K_c^{HW}$, and determine $\sigma$ so that the variances of $V_T$ in the Heston and Hull-White models match.

| | | | | | Heston | | | (matched) Hull-White | |
|---|---|---|---|---|---|---|---|---|---|
| | $T$ | $r$ | $V_0$ | $\rho$ | $\gamma$ | $\theta$ | $\kappa$ | $\mu$ | $\sigma$ |
| Set 1 | 1 | 3.19% | 0.010201 | -0.7 | 0.31 | 0.019 | 6.21 | 1.003 | 0.42 |
| Set 2 | 1/12 | 3.19% | 0.010201 | -0.7 | 0.31 | 0.019 | 6.21 | 4.03 | 1.78 |


Heston Model (T=1)
Hull−White Model (T=1)
Heston Model (T=1/12)
Hull−White Model (T=1/12)
$K_c^H$
$K_c^{HW}$
$K_d^H$
$K_d^{HW}$
ρ = − 0.7
ρ = 0
ρ = 0.7
Discretization step n
0.0188
0.0186
0.0184
0.0182
0.018
0.0178
0.0176
0.0124
0.0123
0.0122
0.0121



Heston Model
Hull−White Model
$K_d^H$ and $K_c^H$
$K_d^{HW}$ and $K_c^{HW}$
$K_d^H$
$K_c^H$
$K_d^{HW}$
$K_c^{HW}$
$V_0$=0.108
$V_0$=0.09
$V_0$=0.072
γ
0.1
0.095
0.09
0.085
0.096
0.094
0.092
0.09
0.088
0.086
0.084
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

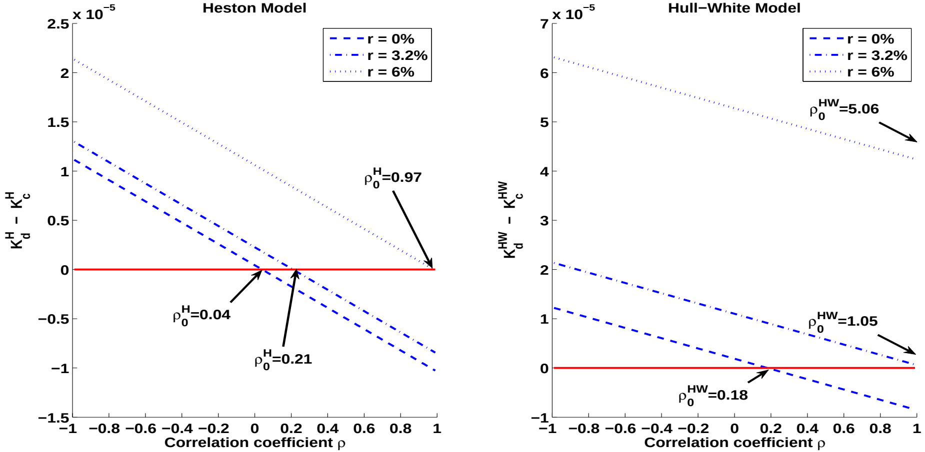



Figure 4: Asymptotic expansion with respect to the correlation coefficient $\rho$ and the risk-free rate $r$

Parameters correspond to Set 1 in Table 1 except for $r$ that can take three possible values $r = 0\%$, $r = 3.2\%$ or $r = 6\%$. Here $n = 250$, which corresponds to a daily monitoring as $T = 1$.

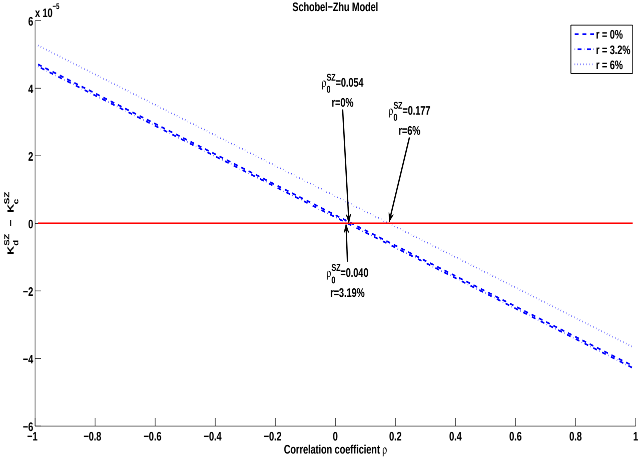



Figure 8: Asymptotic expansion with respect to the correlation coefficient $\rho$ and the risk-free rate $r$.

Parameters are similar to Set 1 in Table 1 for the Heston model except for $r$ that can take three possible values $r = 0\%$, $r = 3.2\%$ or $r = 6\%$. Precisely, we use the following parameters for the Schöbel-Zhu model: $\kappa = 6.21$, $\theta = \sqrt{0.019}$, $\gamma = 0.31$, $\rho = -0.7$, $T = 1$, $V_0 = \sqrt{0.010201}$. Here $n = 250$, which corresponds to a daily monitoring as $T = 1$.

# Conclusions & Future Directions

- Explicit expressions and asymptotics for $K_d^M(n)$ in any time homogeneous stochastic volatility model $(M)$.
- Allow to better understand the effect of discretization.
- Future directions:
  1. Extend our study with the 3/2 model $(dS_t = S_t\sqrt{V_t}dW_1(T), dV_t = (\omega V_t - \theta V_t^2)dt + \xi V_t^{3/2}dW_2(t)$.
  2. Work on expansions valid in a more general setting...
  3. Find out whether the first term in the expansion is always linear in the correlation $\rho$.
  4. Generalize the explicit pricing formula to the case of discrete gamma swaps under the Heston model.

$$Notional \times \frac{1}{T} \times \sum_{i=0}^{n-1} \frac{S_{t_{i+1}}}{S_0}\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}}\right)^2$$

  5. Generalize to the mixed exponential jump diffusion model for which it is possible to compute discrete and continuous fair strikes.

- Bernard, C., and Cui, Z. (2011): Pricing timer options, *Journal of Computational Finance*, 15(1), 69-104.
- Broadie, M., and Jain, A. (2008): The Effect of Jumps and Discrete Sampling on Volatility and Variance Swaps, *IJTAF*, 11, 761-797.
- Bülher, H. (2006): Volatility markets: consistent modeling, hedging and practical implementation , *Ph.D. dissertation, TU Berlin*.
- Cai, N., and S. Kou (2011): Option pricing under a mixed-exponential jump diffusion model, *Management Science*, 57(11), 2067–2081.
- Carr, P., Lee, R., and Wu, L. (2012): Variance swaps on Time-changed Lévy processes, *Finance and Stochastics*, 16(2), 335-355.
- Cui, Z. (2013): PhD thesis at University of Waterloo.
- Hörfelt, P. and Torné, O. (2010): The Value of a Variance Swap - a Question of Interest, *Risk*, June, 82-85.
- Jarrow, R., Y. Kchia, M. Larsson, and P. Protter (2013): "Discretely Sampled Variance and Volatility Swaps versus their Continuous Approximations," *Finance and Stochastics*, 17(2), 305–324.
- Keller-Ressel, M., and C. Griessler (2012): "Convex order of discrete, continuous and predictable quadratic variation and applications to options on variance," *ArXiv Working paper*.
- Keller-Ressel, M., and J. Muhle-Karbe (2012): "Asymptotic and exact pricing of options on variance," *Finance and Stochastics*, forthcoming.