Prices and Asymptotics of Variance Swaps

Carole Bernard
Zhenyu (Rocky) Cui

Beirut, May 2013.
Outline

- Motivation
- Convex order conjecture
- Discrete variance swaps: prices and asymptotics
- Conclusion & Future Directions
A variance swap is an OTC contract:

\[
\text{Notional} \times \left( \frac{1}{T} \, \text{Realized Variance} - \text{Strike} \right)
\]

- Realized Variance: \( \text{RV} = \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2 \right) \) with \( 0 = t_0 < t_1 < \ldots < t_n = T \).

- Quadratic Variation: \( \text{QV} = \lim_{n \to \infty, \max_{i=0,1,\ldots,n-1} (t_{i+1} - t_i) \to 0} \text{RV} \).

- In practice, variance swaps are discretely sampled but it is typically easier to compute the continuously sampled in popular stochastic volatility models.

- Question: Finding “fair” strikes so that the initial value of the contract is 0.
Under the risk-neutral probability measure $Q$, the model setting is given by:

\[
\begin{align*}
\frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)} \\
\quad dV_t &= \mu(V_t)dt + \sigma(V_t)dW_t^{(2)}
\end{align*}
\]

where $\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$. 
Three Stochastic Volatility Models

Assume $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$.

- The **correlated** Heston model:

  \[
  \begin{align*}
  \frac{dS_t}{S_t} &= rdt + \sqrt{V_t} dW_t^{(1)}, \\
  dV_t &= \kappa(\theta - V_t) dt + \gamma \sqrt{V_t} dW_t^{(2)}
  \end{align*}
  \]

- The **correlated** Hull-White model:

  \[
  \begin{align*}
  \frac{dS_t}{S_t} &= rdt + \sqrt{V_t} dW_t^{(1)}, \\
  dV_t &= \mu V_t dt + \sigma V_t dW_t^{(2)}
  \end{align*}
  \]

- The **correlated** Schöbel-Zhu model:

  \[
  \begin{align*}
  \frac{dS_t}{S_t} &= rdt + V_t dW_t^{(1)} \\
  dV_t &= \kappa(\theta - V_t) dt + \gamma dW_t^{(2)}
  \end{align*}
  \]
Model Setting (2/2)

- Under the risk-neutral probability measure $Q$,

\[
\begin{align*}
\frac{dS_t}{S_t} &= rdt + \sqrt{V_t} dW_t^{(1)} \\
\quad dV_t &= \mu(V_t) dt + \sigma(V_t) dW_t^{(2)}
\end{align*}
\]

where $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$.

- The fair strike of the “discrete variance swap” is

\[
K_d^M(n) := \frac{1}{T} \mathbb{E} \left[ \sum_{i=0}^{n-1} \left( \ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2 \right] = \frac{1}{T} \mathbb{E}[RV]
\]

- The fair strike of the “continuous variance swap” is

\[
K_c^M := \frac{1}{T} \mathbb{E} \left[ \int_0^T V_s ds \right] = \frac{1}{T} \mathbb{E}[QV]
\]
Contributions

- A general expression for the fair strike of a discrete variance swap in the time-homogeneous stochastic volatility model:

- Application in three popular stochastic volatility models

- Asymptotic expansion of the fair strike with respect to $n$, $T$, vol of vol...

- A counter-example to the “Convex Order Conjecture”.

Carole Bernard
Convex Order Conjecture

Notations:

1. \( RV = \sum_{i=0}^{n-1} (\log(S_{t_i+1}/S_{t_i}))^2 \): discrete realized variance for a partition of \([0, T]\) with \(n + 1\) points;

2. \( QV = \int_0^T V_s ds \): continuous quadratic variation.

Usual practice: approximate \( \mathbb{E}[f(RV)] \) with \( \mathbb{E}[f(QV)] \), see Jarrow et al (2012).

Bühlher (2006): “while the approximation of realized variance via quadratic variation works very well for variance swaps, it is not sufficient for non-linear payoffs with short maturities”.

Call option on \( RV \): \( (RV - K)^+ \);

Call option on \( QV \): \( (QV - K)^+ \).
Options on Quadratic Variation vs. Realized Variance in a Heston model

Plot from Bühler (2006b).
The convex-order conjecture (Keller-Ressel (2011)):

“The price of a call option on realized variance is higher than the price of a call option on quadratic variation”

Equivalently, \( \mathbb{E}[f(RV)] \geq \mathbb{E}[f(QV)] \) where \( f \) is convex.

When \( f(x) = x \), our closed-form expression shows that when the correlation between the underlying and its variance is positive, it is possible to observe \( K_d^M(n) < K_c^M \) (Illustrated by examples in Heston, Hull-White and Schöbel-Zhu models \((M)\)).
Conditional Black-Scholes Representation

- Recall

\[
\begin{align*}
    \frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)} \\
    dV_t &= \mu(V_t)dt + \sigma(V_t)dW_t^{(2)}
\end{align*}
\]

- Cholesky decomposition: \( dW_t^{(1)} = \rho dW_t^{(2)} + \sqrt{1 - \rho^2}dW_t^{(3)} \).

- Key representation of the log stock price

\[
\ln(S_T) = \ln(S_0) + rt - \frac{1}{2} \int_0^T V_t dt + \rho \left( f(V_T) - f(V_0) - \int_0^T h(V_t) dt \right) + \sqrt{1 - \rho^2} \int_0^T \sqrt{V_t}dW_t^{(3)}
\]

where \( f(v) = \int_0^v \frac{\sqrt{z}}{\sigma(z)} dz, \ h(v) = \mu(v)f'(v) + \frac{1}{2} \sigma^2(v)f''(v) \).
Proposition

Under some technical conditions, \((\Delta = \frac{T}{n})\):

\[
E \left[ \left( \ln \frac{S_{t+\Delta}}{S_t} \right)^2 \right] = r^2 \Delta^2 - r \Delta \int_t^{t+\Delta} E[V_s] \, ds \\
+ \frac{1}{4} E \left[ \left( \int_t^{t+\Delta} V_s \, ds \right)^2 \right] + (1 - \rho^2) \int_t^{t+\Delta} E[V_s] \, ds \\
+ \rho^2 E \left[ (f(V_{t+\Delta}) - f(V_t))^2 \right] + \rho^2 E \left[ \left( \int_t^{t+\Delta} h(V_s) \, ds \right)^2 \right] \\
+ \rho E \left[ \int_t^{t+\Delta} h(V_s) \, ds \int_t^{t+\Delta} V_s \, ds \right] \\
- \rho E \left[ (f(V_{t+\Delta}) - f(V_t)) \int_t^{t+\Delta} (2\rho h(V_s) + V_s) \, ds \right].
\]
**Proposition (Sensitivity to $r$)**

The fair strike of the discrete variance swap:

$$K_d^M(n) = b^M(n) - \frac{T}{n} K_c^M r + \frac{T}{n} r^2,$$

where $b^M(n)$ does **not** depend on $r$.

$$\frac{dK_d^M(n)}{dr} = \frac{T}{n} (2r - K_c^M)$$

$K_d^M(r)$ reaches **minimum** when $r^* = \frac{K_c^M}{2}$. 
Three Stochastic Volatility Models

Assume $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$.

- The **correlated** Heston model:

$\begin{align*}
\text{(H)} & \quad \begin{cases}
\frac{dS_t}{S_t} = r dt + \sqrt{V_t} dW_t^{(1)}, \\
dV_t = \kappa(\theta - V_t) dt + \gamma \sqrt{V_t} dW_t^{(2)}
\end{cases}
\end{align*}$

- The **correlated** Hull-White model:

$\begin{align*}
\text{(HW)} & \quad \begin{cases}
\frac{dS_t}{S_t} = r dt + \sqrt{V_t} dW_t^{(1)}, \\
dV_t = \mu V_t dt + \sigma V_t dW_t^{(2)}
\end{cases}
\end{align*}$

- The **correlated** Schöbel-Zhu model:

$\begin{align*}
\text{(SZ)} & \quad \begin{cases}
\frac{dS_t}{S_t} = r dt + V_t dW_t^{(1)} \\
dV_t = \kappa(\theta - V_t) dt + \gamma dW_t^{(2)}
\end{cases}
\end{align*}$
The fair strike of the **discrete** variance swap is

\[
K_d^H(n) = \frac{1}{8n\kappa^3 T} \left\{ 2\kappa T \left( \kappa^2 T (\theta - 2r)^2 + n\theta \left( 4\kappa^2 - 4\rho\kappa\gamma + \gamma^2 \right) \right) \\
+ n \left( \gamma^2 (\theta - 2V_0) + 2\kappa (V_0 - \theta)^2 \right) \left( e^{-2\kappa T} - 1 \right) \frac{1 - e^{\frac{\kappa T}{n}}}{1 + e^{\frac{\kappa T}{n}}} \\
+ 4 (V_0 - \theta) \left( n (2\kappa^2 + \gamma^2 - 2\rho\kappa\gamma) + \kappa^2 T (\theta - 2r) \right) \left( 1 - e^{-\kappa T} \right) \\
- 2n^2\theta\gamma (\gamma - 4\rho\kappa) \left( 1 - e^{-\frac{\kappa T}{n}} \right) + 4 (V_0 - \theta) \kappa T \gamma (\gamma - 2\rho\kappa) \frac{1 - e^{-\kappa T}}{1 - e^{\frac{\kappa T}{n}}} \right\}
\]

The fair strike of the **continuous** variance swap is

\[
K_c^H = \frac{1}{T} \mathbb{E} \left[ \int_0^T V_s ds \right] = \theta + (1 - e^{-\kappa T}) \frac{V_0 - \theta}{\kappa T}.
\]
The fair strike of the **discrete** variance swap is

\[
K_{d}^{HW}(n) = \frac{r^2 T}{n} + \frac{V_0}{\mu T} \left( 1 - \frac{rT}{n} \right) \left( e^{\mu T} - 1 \right)
- \frac{V_0^2 \left( e^{(2\mu + \sigma^2)T} - 1 \right) \left( e^{\frac{\mu T}{n}} - 1 \right)}{2T \mu (\mu + \sigma^2) \left( e^{\frac{(2\mu + \sigma^2)T}{n}} - 1 \right)} + \frac{V_0^2 \left( e^{(2\mu + \sigma^2)T} - 1 \right)}{2T (2\mu + \sigma^2) (\mu + \sigma^2)}
+ \frac{8\rho \left( e^{\frac{3(4\mu + \sigma^2)T}{8}} - 1 \right) V_0^{3/2} \sigma \left( e^{\frac{\mu T}{n}} - 1 \right)}{\mu T (4\mu + 3\sigma^2) \left( e^{\frac{3(4\mu + \sigma^2)T}{8n}} - 1 \right)} - \frac{64\rho \left( e^{\frac{3(4\mu + \sigma^2)T}{8}} - 1 \right) V_0^{3/2} \sigma}{3T (4\mu + \sigma^2) (4\mu + 3\sigma^2)}
\]

The fair strike of the **continuous** variance swap is

\[
K_{c}^{HW} = \frac{1}{T} \mathbb{E} \left[ \int_0^T V_s ds \right] = \frac{V_0}{T\mu} (e^{\mu T} - 1).
\]
The fair strike of the \textbf{discrete} variance swap is explicit but too complicated to appear on a slide. The fair strike of the \textbf{continuous} variance swap is

\[
K_{c}^{SZ} = \frac{\gamma^2}{2\kappa} + \theta^2 + \left( \frac{(V_0 - \theta)^2}{2\kappa T} - \frac{\gamma^2}{4\kappa^2 T} \right) \left(1 - e^{-2\kappa T}\right) + \frac{2\theta(V_0 - \theta)}{\kappa T} \left(1 - e^{-\kappa T}\right).
\]
Heston model: Expansion w.r.t $n$

\[ K^H_d(n) = K^H_c + \frac{a^H_1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right). \]

where $a^H_1$ is a **linear and decreasing** function of $\rho$:

\[ a^H_1 \geq 0 \iff \rho \leq \rho^H_0 \]

where

\[ \rho^H_0 = \frac{r^2 T - rK^H_c T + \left(\frac{\theta^2}{4} + \frac{\theta \gamma^2}{8\kappa}\right) T + c_1}{\left(\frac{\gamma(\theta-V_0)}{2\kappa}(1 - e^{-\kappa T}) - \frac{\theta \gamma T}{2}\right)}. \]

\[ ^1 \text{Explicit expression of } a^H_1 \text{ is in Proposition 5.1, Bernard and Cui (2012).} \]
Hull-White model: Expansion w.r.t $n$

$$K_d^{HW}(n) = K_c^{HW} + \frac{a_1^{HW}}{n} + O\left(\frac{1}{n^2}\right)$$

where $a_1^{HW}$ is a linear and decreasing function of $\rho$:²

$$a_1^{HW} \geq 0 \iff \rho \leq \rho_0^{HW}$$

where

$$\rho_0^{HW} = \frac{3(4\mu + \sigma^2) \left( r^2 T - r K_c^{HW} T + \frac{V_0^2}{4} \frac{e^{(2\mu+\sigma^2)T} - 1}{2\mu+\sigma^2} \right)}{4\sigma V_0^2 \left( e^{3\frac{3}{8}(4\mu+\sigma^2)T} - 1 \right)} > 0.$$
The asymptotic behavior of the fair strike of a discrete variance swap in the Schöbel-Zhu model is given by

\[ K_{d\,SZ}^S(n) = K_{c\,SZ} + \frac{a_{1\,SZ}}{n} + O\left(\frac{1}{n^2}\right), \]

where

\[ a_{1\,SZ} = r^2 T - rTK_{c\,SZ} + d_1 + d_2 \frac{\gamma}{2\kappa} \rho. \] (1)

and where \( d_1 \) and \( d_2 \) are explicit.
Given that the expressions are explicit, it is straightforward to obtain expansions for the discrete variance swaps as a function of the different parameters, and for example with respect to the maturity or to the volatility of volatility.
Expansion of the fair strike for small maturity $T$

In the **Heston model**, an expansion of $K_d^H(n)$ when $T \to 0$ is

$$K_d^H(n) = V_0 + b_1^H T + b_2^H T^2 + \mathcal{O}(T^3)$$

where

$$b_1^H = \frac{\kappa(\theta - V_0)}{2} + \frac{1}{4n} \left( (V_0 - 2r)^2 - 2\gamma V_0\rho \right)$$

$$b_2^H = \frac{\kappa^2(V_0 - \theta)}{6} + \frac{(V_0 - \theta)\kappa(\gamma\rho + 2r - V_0) + \gamma^2 V_0}{4n} + \frac{\gamma\rho\kappa(V_0 + \theta) - \gamma^2 V_0}{12n^2}.$$ 

and we have

$$K_d^H(n) - K_c^H = \frac{1}{4n} \left( (V_0 - 2r)^2 - 2\rho\gamma V_0 \right) T + \mathcal{O}(T^2).$$
Expansion of the fair strike for small maturity

In the **Hull-White model**, an expansion of $K_{d}^{HW}(n)$ when $T \to 0$ is

$$K_{d}^{HW}(n) = V_0 + b_{1}^{HW} T + b_{2}^{HW} T^2 + O(T^3)$$

where

$$b_{1}^{HW} = \frac{V_0 \mu}{2} + \frac{1}{4n} \left( (V_0 - 2r)^2 - 2\rho V_0^{3/2} \right)$$

$$b_{2}^{HW} = \frac{V_0 \mu^2}{6} + \frac{V_0}{4n} \left( \frac{\sigma^2 V_0}{2} - \frac{3\rho V_0^{1/2} \sigma (\sigma^2 + 4\mu)}{8} + \mu (V_0 - 2r) \right)$$

$$+ \frac{V_0^{3/2} \sigma (\rho (3\sigma^2 - 4\mu) - 4\sigma \sqrt{V_0})}{96n^2}$$

Note also

$$K_{d}^{HW}(n) - K_{c}^{HW} = \frac{1}{4n} \left( (V_0 - 2r)^2 - 2\rho V_0^{3/2} \right) T + O(T^2).$$
Expansion of the fair strike for small maturity

In the **Schöbel-Zhu model**, an expansion of $K_d^{HW}(n)$ when $T \to 0$ is

$$K_d^{SZ}(n) = V_0^2 + b_1^{SZ} T + O(T^2)$$

where

$$b_1^{SZ} = \kappa V_0 (\theta - V_0) + \frac{\gamma^2}{2} + \frac{1}{n} \left( r^2 - r V_0^2 + \frac{V_0^2 (V_0^2 - 4 \rho \gamma)}{4} \right)$$

Note also

$$K_d^{SZ}(n) - K_c^{SZ} = \frac{1}{4n} \left( (V_0^2 - 2r)^2 - 4 \rho V_0^2 \gamma \right) T + O(T^2).$$
Parameters

- **Heston model**: First set of parameters from Broadie and Jain (2008). Second set is when $T = 1/12$.

- **Hull-White model**: obtain $\mu$ by numerically solving $K_c^H = K_c^{HW}$, and determine $\sigma$ so that the variances of $V_T$ in the Heston and Hull-White models match.

<table>
<thead>
<tr>
<th></th>
<th>$T$</th>
<th>$r$</th>
<th>$V_0$</th>
<th>$\rho$</th>
<th>$\gamma$</th>
<th>$\theta$</th>
<th>$\kappa$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set 1</td>
<td>1</td>
<td>3.19%</td>
<td>0.010201</td>
<td>-0.7</td>
<td>0.31</td>
<td>0.019</td>
<td>6.21</td>
<td>1.003</td>
<td>0.42</td>
</tr>
<tr>
<td>Set 2</td>
<td>1/12</td>
<td>3.19%</td>
<td>0.010201</td>
<td>-0.7</td>
<td>0.31</td>
<td>0.019</td>
<td>6.21</td>
<td>4.03</td>
<td>1.78</td>
</tr>
</tbody>
</table>
Heston Model (T=1)

- $K_H^d$
- $\rho = -0.7$
- $\rho = 0$
- $\rho = 0.7$

Hull–White Model (T=1)

- $K_{HW}^d$
- $\rho = -0.7$
- $\rho = 0$
- $\rho = 0.7$

Heston Model (T=1/12)

- $K_H^d$
- $\rho = -0.7$
- $\rho = 0$
- $\rho = 0.7$

Hull–White Model (T=1/12)

- $K_{HW}^d$
- $\rho = -0.7$
- $\rho = 0$
- $\rho = 0.7$
Figure 4: Asymptotic expansion with respect to the correlation coefficient $\rho$ and the risk-free rate $r$

Parameters correspond to Set 1 in Table 1 except for $r$ that can take three possible values $r = 0\%$, $r = 3.2\%$ or $r = 6\%$. Here $n = 250$, which corresponds to a daily monitoring as $T = 1$. 
Figure 8: Asymptotic expansion with respect to the correlation coefficient $\rho$ and the risk-free rate $r$.
Parameters are similar to Set 1 in Table 1 for the Heston model except for $r$ that can take three possible values $r = 0\%$, $r = 3.2\%$ or $r = 6\%$. Precisely, we use the following parameters for the Schöbel-Zhu model: $\kappa = 6.21$, $\theta = \sqrt{0.019}$, $\gamma = 0.31$, $\rho = -0.7$, $T = 1$, $V_0 = \sqrt{0.010201}$. Here $n = 250$, which corresponds to a daily monitoring as $T = 1$. 
Conclusions & Future Directions

- Explicit expressions and asymptotics for $K_d^M(n)$ in any time homogeneous stochastic volatility model ($M$).
- Allow to better understand the effect of discretization.
- Future directions:
  1. Extend our study with the 3/2 model
     \[
     (dS_t = S_t \sqrt{V_t} dW_1(t), \quad dV_t = (\omega V_t - \theta V_t^2) dt + \xi V_t^{3/2} dW_2(t)).
     \]
  2. Work on expansions valid in a more general setting...
  3. Find out whether the first term in the expansion is always linear in the correlation $\rho$.
  4. Generalize the explicit pricing formula to the case of discrete gamma swaps under the Heston model.

\[
Notional \times \frac{1}{T} \times \sum_{i=0}^{n-1} \frac{S_{t_{i+1}}}{S_0} \left( \ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2
\]

5. Generalize to the mixed exponential jump diffusion model for which it is possible to compute discrete and continuous fair strikes.
<table>
<thead>
<tr>
<th>Motivation</th>
<th>Convex Order Conjecture</th>
<th>Variance Swap</th>
<th>Numerics</th>
<th>Conclusions</th>
</tr>
</thead>
</table>