Financial Bounds for Insurance Claims

Steven Vanduffel (Vrije Universiteit Brussel)

Carole Bernard (University of Waterloo)

1 Motivation

To find bounds:

- for the price p of a future insurance claim C_T that cannot be hedged.

- but for which we know the cdf F_{C_T} under the physical probability measure $\mathbb{P}$.

Most premium principles appearing in the literature satisfy the positive loading condition (no-undercut) i.e.

$$p \geq E[C_T]e^{-rT},$$

where r is the fixed rate one is earning in $(0, T)$. (see e.g. Embrechts, 2000).

2 Main result

We challenge:

$$\text{Traditional lowerbound} := E[C_T]e^{-rT}.$$

We argue it need to be corrected to reflect possible interaction with the financial market:

$$\text{New lowerbound} := E[C_T]e^{-rT} + \mathbf{Cov(C_T, \boldsymbol{\xi}_T)},$$

where ξ_T is some financial payoff (we specify this further).

3 Assumptions on Preferences

1. Agents have a fixed investment horizon $T > 0$.

2. Agents have "law-invariant" preferences. i.e. for an objective function $V(.)$ and final wealths $X_T \sim Y_T$ it holds that $V(X_T) = V(Y_T)$.

3. Agents prefer "more to less": for a non-negative r.v. C: $V(X_T + C) \geq V(X_T)$.

4. Agents are risk-averse:

$$\begin{cases} E[X_T] = E[Y_T] \\ \forall d \in \mathbb{R}, E[(X_T - d)^+] \leq E[(Y_T - d)^+] \end{cases} \Rightarrow V(X_T) \geqslant V(Y_T).$$

4 Traditional approach: certainty equivalents

Setting:

- From the **viewpoint of the insured** with objective function $U(\cdot)$ and initial wealth ω_b the (bid) price p^b follows from:

$$U[(\omega_b - p^b)e^{rT}] = U[\omega_b e^{rT} - C_T].$$

- From the **viewpoint of the insurer** with objective function $V(\cdot)$ and initial wealth ω_a the ask price p^a follows from:

$$V[(\omega_a + p^a)e^{rT} - C_T] = V[\omega_a e^{rT}].$$

  Note that the financial market only appears through the availability of a bank account earning the fixed intrest rate $r > 0$.

Properties:

- Bid and Ask prices verify the **no-undercut** principle:

$$p^{\cdot} \geqslant e^{-rT} E[C_T],$$

where we have used notation $p^{\cdot}$ to reflect both p^a and p^b

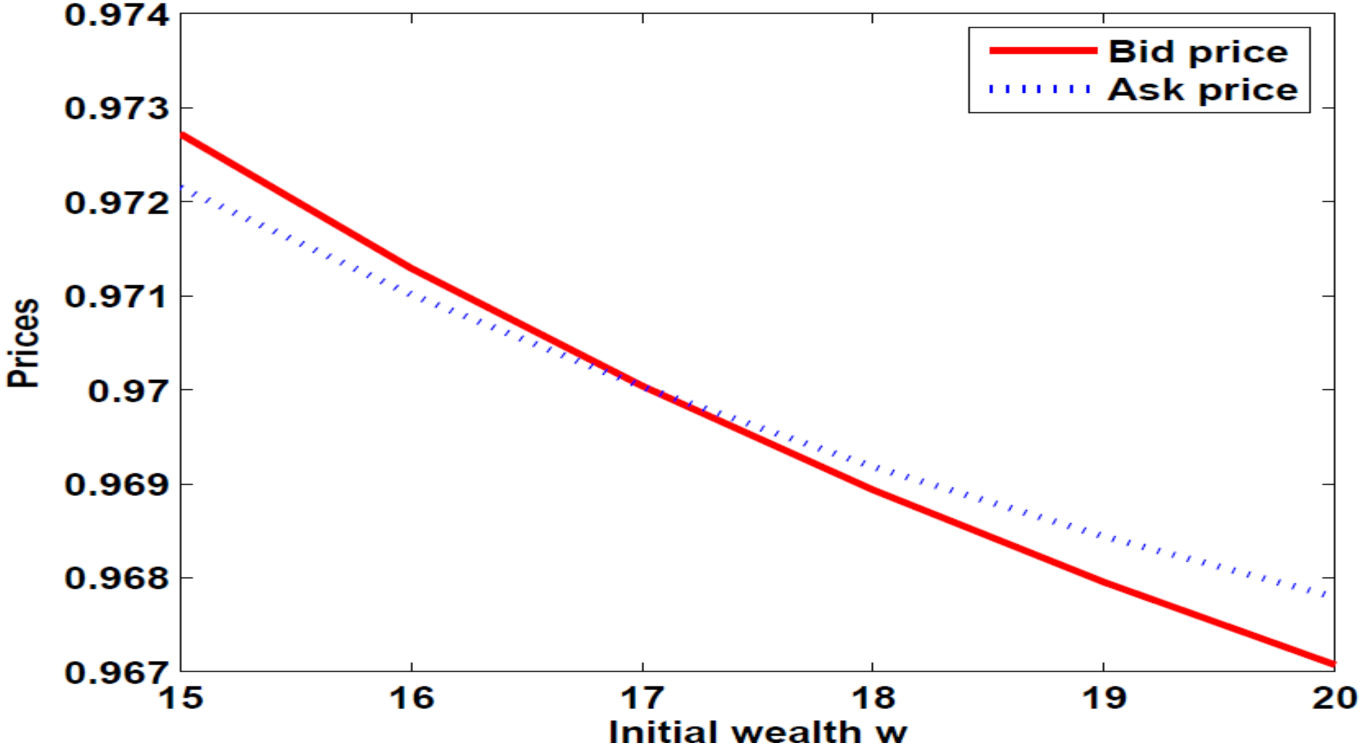
- If the insurer is risk neutral ($v(x) = x$), then

$$p_b \geq p_a = e^{-rT} E[C_T].$$

- In the case of exponential utility (when $U = V$) $p_a = p_b$.

- In the case of Yaari's theory (when $U = V$) $p_a = p_b$.

In general, no strong assertions regarding the ordering between p_a and p_b are in reach. Assume $u(x) = v(x) = 1 - 1/x$ and let both agents have the same initial wealth, $C_T \sim U(0, 2)$:



Issue:

- This framework ignores completely the available prices of other financial instruments and one may then already wonder if it can possibly be used to price claims that are connected with the financial market.

- Indeed this framework is incompatible with pricing of financial claims. Assume a common stock with payoff S_T at time T. The price S_0 is usually such that $\mathsf{E}(S_T) > S_0 e^{rT}$. Hence

$$S_0 < e^{-rT} E[S_T],$$

in other words we violate the traditional lower bound.

Questions:

- How to integrate the presence of financial markets in the framework of certainty equivalents.
- Can we ensure that the resulting pricing mechanism is coherent with the prices of financial instruments.
- What is the impact of the new framework, if any, on the stated classical lower bound.

5 Financial pricing

Assumption: There is a financial (sub) market containing a riskless asset and a risky asset S such that all call options (written on S) maturing at time $T > 0$ are traded.

Consequence: There is a (so-called risk neutral) measure Q such that for all claims $X_T = f(S_T)$ it holds that

$$p_a = p_b = e^{-rT} E_Q[X_T],$$

or equivalently, there is payoff ξ_T such that the price of a financial claim X_T can also be expressed as

$$p_a = p_b = E_P[\xi_T X_T],$$

6 A market consistent approach

- Let $A(w)$ be the set of random financial wealths X_T that can be obtained (in the financial sub-market) for the initial budget $w > 0$. From the **viewpoint of the insured** with objective function $U(\cdot)$ and initial wealth ω_b the (bid) price p^b follows from:

$$\sup_{X_T \in A(w_b - p^b)} \{U[X_T]\} = \sup_{X_T \in A(w_b)} \{U[X_T - C_T]\}.$$

- From the **viewpoint of the insurer** with objective function $V(\cdot)$ and initial wealth w_a the ask price p^a follows from:

$$\sup_{X_T \in A(w_a + p^a)} \{V[X_T - C_T]\} = \sup_{X_T \in A(w_a)} \{V[X_T]\}.$$

(see e.g. Hodges and Neuberger (1989) or also Henderson & Hobson (2004))

Properties:

• This approach can be shown to be **market consistent**, i.e. when C_T is a financial claim then one has that $p^b = p^a = E[\xi_T.C_T]$.

• In general computing the bid and ask prices p^b and p^a explicitly is not in reach (in the paper we show how the technique of pathwise optimisation can be helpful).

• This stresses the need for determining bounds that can be computed easily.

7 New Lower bound

• We find that

$$\mathbf{p}^{\cdot} \geq \mathbf{E}[\boldsymbol{\xi}_T.\mathbf{C}_T].$$

• Hence both the insured and the insurer are potentially prepared to agree on a price for the **insurance payoff** C_T which is larger than the price "like if C_T would be a **financial payoff"**.

• This result is rooted in work on cost-efficient financial payoffs (Bernard, Boyle and Vanduffel, 2011).

• Remark that the lower bound E[$\xi_T.C_T$] is actually the market price of the financial payoff $E[C_T|\xi_T]$.

• We then also find that

$$p^{\cdot} \geq e^{-rT}.E[C_T]+\mathbf{Cov}[\mathbf{C}_T, \boldsymbol{\xi}_T].$$

• Hence when the claim C_T and the state-price ξ_T are negatively correlated we find that $e^{-rT}.E[C_T]$ is no longer a lower bound for p^b and p^a, which contrasts with traditional (and intuitively appealing) wisdom stated in many actuarial text books.

• Note that if we only allow for the riskless asset to exist, then $A(w) = \left\{we^{rT}\right\}, \xi_T = e^{-rT}$ and we obtain the traditional lowerbound $e^{-rT}.E[C_T]$ again.

- **If C_T is independent of (the market) $\boldsymbol{\xi}_T$,**

$$p^{\cdot} \geq e^{-rT}.E[C_T].$$

The independence implies that the financial market cannot help at all to hedge the insurance claim. It appears therefore intuitive that our bound coincides with the classical bound.

- **If C_T is positively correlated with the market**, the classical lower bound $e^{-rT}E[C_T]$ is now strictly improved.

$$p^{\cdot} \geq e^{-rT}.E[C_T] + Cov[C_T, \xi_T] > e^{-rT}.E[C_T].$$

- However **if C_T is negatively correlated with the market**, the lower bound is smaller

$$p^{\cdot} \geq e^{-rT}.E[C_T] + Cov[C_T, \xi_T].$$

E.g. The best lower bound for equity-linked insurance benefits will generally be lower than $e^{-rT}E[C_T]$ because

$$\begin{aligned} Cov(S_T, \xi_T) &= E[S_T\xi_T] - E[S_T]E[\xi_T] \\ &= e^{-rT}(E_Q[S_T] - E_P[S_T]), \end{aligned}$$

8 Example

In the Black-Scholes model,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^{\mathbb{P}},$$

with $\mu > r$. The state price process exists and is unique $\xi_t = a\left(\frac{S_t}{S_0}\right)^{-\frac{\theta}{\sigma}}$, where $a = e^{\frac{\theta}{\sigma}(\mu - \frac{\sigma^2}{2})t - (r + \frac{\theta^2}{2})t}$ and $\theta = \frac{\mu - r}{\sigma}$.

Note that ξ_t is decreasing in S_t, then for all $c \in \mathbb{R}$

$$\mathbb{P}(S_t > c) > \mathbb{Q}(S_t > c),$$

Consider a very simple insurance claim that pays at time $T = 1$ a payoff C_1 distributed as a Bernoulli r.v.

3 cases:

First, the insurance claim C_1 is linked to the death of a specific individual, then

$$\mathsf{E}[C_1|\xi_1] = \mathsf{E}[C_1].$$

and

$$\mathsf{E}[C_1] = \mathbb{P}(death).$$

Bid and ask prices $p^{\cdot}$ satisfy

$$p^{\cdot} \geq E[\xi_1 \mathsf{E}[C_1|\xi_1]] = e^{-r}E[C_1] = e^{-r}\mathbb{P}(death).$$

Second, C_1 pays 1 if the individual dies and the risky asset in the market is higher than a value H or equivalently $\{\xi_1 < L\} = \{S_1 > H\}$). Then

$$\begin{aligned} \mathsf{E}[C_1|\xi_1] &= \mathsf{E}[1_{death}1_{\xi_1<L}|\xi_1] \\ &= \mathbb{P}(death)1_{S_1>H}. \end{aligned}$$

and

$$\mathsf{E}[C_1] = \mathbb{P}(death)\mathbb{P}(S_1 > H).$$

Then bid and ask prices need to satisfy

$$p^{\cdot} \geq e^{-r}.\mathbb{P}(death)\mathbb{Q}(S_1 > H),$$

and we violate the classical lower bound

$$e^{-r}.\mathbb{P}(death)\mathbb{Q}(S_1 > H) < e^{-r}E[C_1].$$

Third, C_1 pays 1 if a designated person dies and the risky asset in the market is lower than a value H. Then, $Cov(C_1, \xi_1) > 0$ and bid and ask prices satisfy

$$p^{\cdot} \geq e^{-r}.\mathbb{P}(death).\mathbb{Q}(S_1 < H)$$

and we improve the classical lower bound

$$e^{-r}.\mathbb{P}(death)\mathbb{Q}(S_1 < H) > e^{-r}E[C_1].$$

9 Final Remarks

• We have determined a lower bound for the price of an insurance claim, and it corresponds to the price of some financial payoff. Note that if we have a financial market with the riskless asset only we obtain the classical lower bound again.

• The new lower bound is not restricted to EUT setting.

• In the paper we also discuss partial insurance. Some but not all results continue to hold.

• In the paper we also introduce another lower bound under a much milder notion of risk aversion.

10 References

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11 Additional material on cost-efficiency

11.1 Set-up

• Consider an arbitrage-free and complete financial market with a corresponding probability space $(\Omega, \mathcal{F}, P)$.

• Given a strategy with payoff X_T at time $T > 0$. There exists a measure Q such that its price at 0 is

$$c(X_T) = E_Q[e^{-rT} X_T].$$

• P ("physical measure") and Q ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT}(\frac{dQ}{dP})_T,$$

and the cost $c(X_T)$ also writes as

$$c(X_T) = E\left[\xi_T X_T\right].$$

• We assume ξ_T is continuously distributed.

11.2 Some Results

- Same distribution - lower cost (Bernard, Boyle, Vanduffel (2011))

The solution for

$$\underset{\{X_T \mid X_T \sim G\}}{\text{Min}} c\{X_T\}$$

is given by $X_T^* = h(\xi_T)$ with $h(\cdot) = G^{-1}(1 - F_{\xi_T}(\cdot))$.

Proof

X_T^* has distribution G. It is also anti-monotonic with ξ_T. Hence amongst all payoffs with fixed distribution G, it is X_T^* which has minimal correlation with ξ_T, or equivalently, the cost $c(X_T^*) = E[\xi_T X_T^*]$ is minimal.

• Same cost - less spread (Bernard, Boyle, Vanduffel (2011))

The payoff $E\left[X_T|\xi_T\right]$has the same cost as X_T (but has less spread).

Proof We have that

$$\begin{aligned} c(X_T) &= E[\xi_T.X_T] \\ &= E[E[\xi_T.X_T|\xi_T] \\ &= E[\xi_T.E[X_T|\xi_T]] \\ &= c(E[X_T|\xi_T]). \end{aligned}$$

• Both results allow to find optimal strategies for investors who only care about the distribution of final wealth.