Explicit Representation of Cost-efficient Strategies

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joint work with

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Contributions

- Deriving **explicitly** the cheapest and the most expensive strategy to achieve a given distribution under general assumptions on the financial market.
- Extension of the work by
  Cox, J.C., Leland, H., 1982. "*On Dynamic Investment Strategies*," *Proceedings of the seminar on the Analysis of Security Prices*, U. of Chicago. (published in 2000 in *JEDC*).
  Dybvig, P., 1988a. "*Distributional Analysis of Portfolio Choice*," *Journal of Business*.
  Dybvig, P., 1988b. "*Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market*," *RFS*.
- Suboptimality of path-dependent contracts in Black Scholes model

Some Assumptions

- Consider an arbitrage-free and complete market.
- Given a strategy with payoff X_T at time T. There exists Q, such that its price at 0 is

$$P_X = E_Q[e^{-rT} X_T]$$

- P ("physical measure") and Q ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT} \left(\frac{dQ}{dP}\right)_T, \quad P_X = E_Q[e^{-rT} X_T] = E_P[\xi_T X_T].$$

Motivation: Traditional Approach to Portfolio Selection

Investors have a strategy that will give them a final wealth X_T.
This strategy depends on the financial market and is random.

- For example they want to maximize the **expected utility** of their final wealth X_T

$$\max_{X_T} \left(E_P[U(X_T)]\right)$$

U: utility (increasing because individuals prefer more to less).

- for a given **cost of the strategy**

$$cost\ at\ 0 = E_Q[e^{-rT}X_T] = E_P[\xi_T X_T]$$

Find optimal payoff X_T $\Rightarrow$ Optimal cdf F of X_T

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Cost-efficient strategies

- Given the cdf F that the investor would like for his final wealth
- We derive an explicit representation of the payoff X_T such that
  - $X_T \sim F$ in the real world
  - X_T corresponds to the cheapest strategy (=cost-efficient strategy)

- What is cost-efficiency?
- **Explicit** construction of **cost-efficient** strategies.

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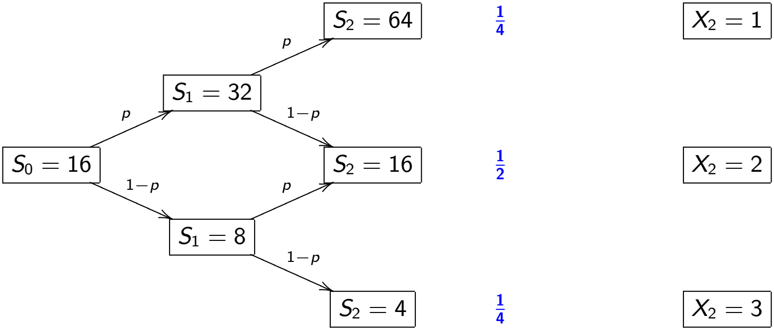
A Simple Illustration

Let's illustrate what the "efficiency cost" is with a simple example. Consider :

- A market with 2 assets: a bond and a stock S.
- A discrete 2-period binomial model for the stock S.
- A strategy with payoff X_T at the end of the two periods.
- An expected utility maximizer with utility function U.

A simple illustration for X_2, a payoff at $T=2$

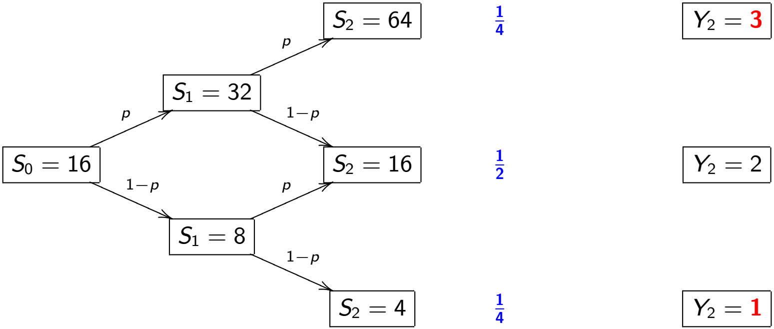
Real-world probabilities$=p=\frac{1}{2}$



$$E[U(X_2)]=\frac{U(1)+U(3)}{4}+\frac{U(2)}{2}$$

Y_2, a payoff at $T = 2$ distributed as X_2

Real-world probabilities$=p = \frac{1}{2}$



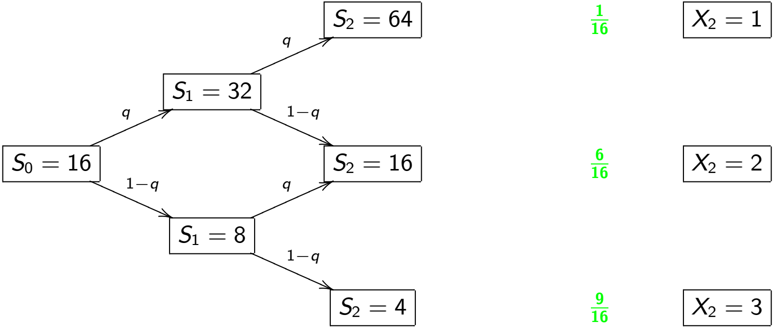
$$E[U(Y_2)] = \frac{U(3) + U(1)}{4} + \frac{U(2)}{2}$$

(X and Y have the same distribution under the physical measure and thus the same utility)

X_2, a payoff at $T = 2$

risk neutral

probabilities$=q = \frac{1}{4}$.



$$P_{X_2} = \text{Price of } X_2 = \left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2}$$

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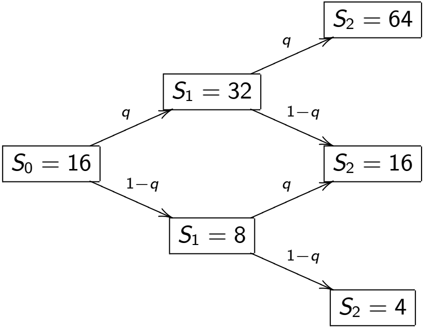
risk neutral

probabilities$=q = \frac{1}{4}$.



$\frac{1}{16}$	$\mathbf{Y_2 = 3}$
$\frac{6}{16}$	$Y_2 = 2$
$\frac{9}{16}$	$\mathbf{Y_2 = 1}$

$$P_{Y_2} = \left(\frac{1}{16}3 + \frac{6}{16}2 + \frac{9}{16}1\right) = \frac{3}{2}$$

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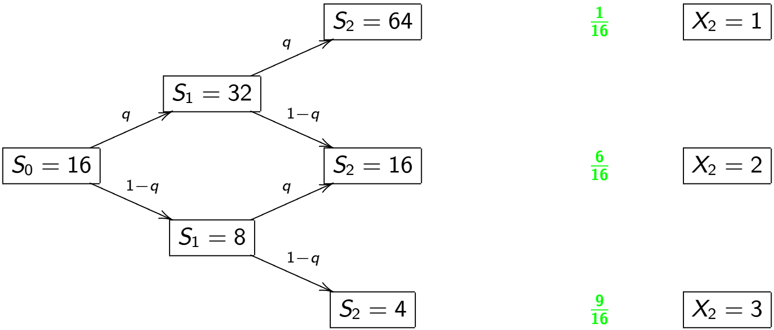
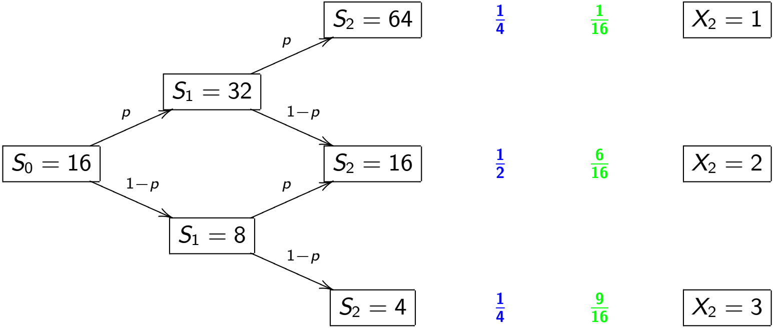


$$P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} = \text{Price of } X_2 = \frac{5}{2} \quad , \quad \text{Efficiency cost} = P_{X_2} - P_D$$

A simple illustration for X_2, a payoff at $T = 2$

Real-world probabilities$=p = \frac{1}{2}$ and **risk neutral** probabilities$=q = \frac{1}{4}$.



$$E[U(X_2)] = \frac{U(1) + U(3)}{4} + \frac{U(2)}{2} \quad , \quad P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} \ = \ \text{Price of } X_2 = \frac{5}{2} \quad , \quad \text{Efficiency cost} = P_{X_2} - P_D$$

Efficiency Cost

- Given a strategy with payoff X_T at time T, and initial price at time 0
$$P_X = E_P\left[\xi_T X_T\right]$$
- F : X_T's distribution under the **physical measure** P.

The distributional price is defined as

$$PD(F) = \min_{\{Y_T \mid Y_T \sim F\}} \{E_P\left[\xi_T Y_T\right]\} = \min_{\{Y_T \mid Y_T \sim F\}} c(Y_T)$$

The "loss of efficiency" or "efficiency cost" is equal to:

$$P_X - PD(F)$$

Criteria for evaluating payoffs independent of the agents' preferences.

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Minimum Price = Cost-efficiency

Theorem

Consider the following optimization problem:

$$\min_{\{Z \mid Z \sim F\}} \{c(Z)\}$$

Assume ξ_T *is continuously distributed, then the optimal strategy is*

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

Note that $X_T^\star \sim F$ *and* $X_T^\star$ *is a.s. unique such that*

$$PD(F) = c(X_T^\star)$$

Thanks to the uniqueness, we characterize all cost-efficient strategies.

Black and Scholes Model

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Under the risk neutral measure Q,

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^Q$$

$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T = e^{-rT} a \left(\frac{S_T}{S_0}\right)^{-b}$ where a and b are positive and constant.

Any path-dependent financial derivative is inefficient.
To be cost-efficient, the contract has to be a European derivative written on S_T and non-decreasing w.r.t. S_T (when $\mu \geqslant r$). In this case,

$$X^\star = F^{-1}\left(F_S\left(S_T\right)\right)$$

Geometric Asian contract in Black and Scholes model

Assume a strike K. The payoff of the Geometric Asian call is given by

$$G_T = \left(e^{\frac{1}{T}\int_0^T \ln(S_t)dt} - K\right)^+$$

which corresponds in the discrete case to $\left(\left(\prod_{k=1}^n S_{\frac{kT}{n}}\right)^{\frac{1}{n}} - K\right)^+$.

The efficient payoff that is distributed as the payoff G_T is given by

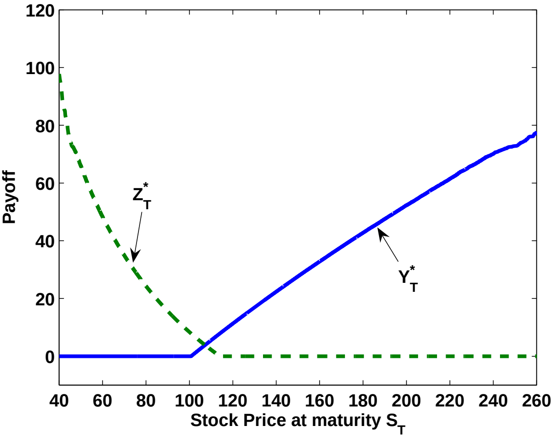
$$G_T^\star = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

where $d := S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$.

This payoff $G_T^\star$ is a power call option. If $\sigma = 20\%, \mu = 9\%$, $r = 5\%$, $S_0 = 100$. The price of this geometric Asian option is 5.94. The payoff $G_T^\star$ costs only 5.77.

Similar result in the discrete case.

Example: the discrete Geometric option



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100,\ n = 12$.

Price of the geometric Asian option = 5.94. The distributional price is 5.77.

The least-efficient payoff $Z_T^\star$ costs 9.03.

Put option in Black and Scholes model

Assume a strike K. The payoff of the put is given by

$$L_T = (K - S_T)^+ .$$

The payoff that has the **lowest** cost and is distributed such as the put option is given by

$$Y_T^\star = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

Put option in Black and Scholes model

Assume a strike K. The payoff of the put is given by

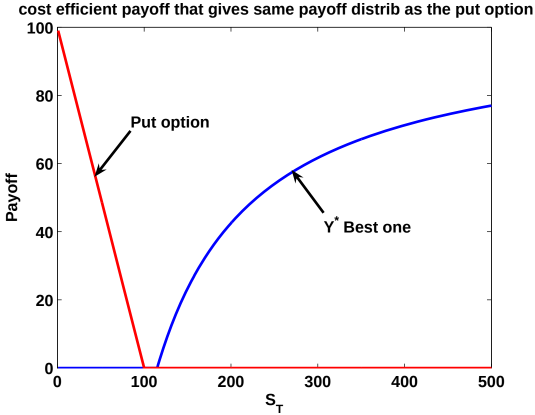
$$L_T = (K - S_T)^+ .$$

The cost-efficient payoff that will give the same distribution as a put option is

$$Y_T^\star = \left(K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T} \right)^+ .$$

This type of power option "dominates" the put option.

Cost-efficient payoff of a put



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional price of the put = 3.14

Price of the put = 5.57

Efficiency loss for the put = 5.57-3.14= 2.43

Utility Independent Criteria

Denote by

- X_T the final wealth of the investor,
- $V(X_T)$ the objective function of the agent,

Assumptions

1. **Agents' preferences depend only on the probability distribution of terminal wealth:** "law-invariant" preferences. (if $X_T \sim Z_T$ then: $V(X_T) = V(Z_T)$.)
2. **Agents prefer "more to less"**: if c is a non-negative random variable $V(X_T + c) \geqslant V(X_T)$.
3. The market is perfectly liquid, no taxes, no transaction costs, no trading constraints (in particular short-selling is allowed).
4. The market is **arbitrage-free** and **complete**.

Any optimal investment has to be cost-efficient.

Explaining the Demand for Inefficient Payoffs

1. State-dependent needs
   - **Background risk:**
     - Hedging a long position in the market index S_T (background risk) by purchasing a put option P_T,
     - the background risk can be path-dependent.
   - **Stochastic benchmark or other constraints:** If the investor wants to outperform a given (stochastic) benchmark Γ such that:
     $$P\left\{\omega \in \Omega \,/\, W_T(\omega) > \Gamma(\omega)\right\} \geqslant \alpha.$$
   - **Intermediary consumption.**
2. **Other sources of uncertainty:** Stochastic interest rates or stochastic volatility
3. **Transaction costs, frictions**

Conclusions

- A preference-free framework for ranking different investment strategies.
- For a given investment strategy, we derive an explicit analytical expression
  1. for the cheapest strategy that has the same payoff distribution.
  2. for the most expensive strategy that has the same payoff distribution.
- There are strong connections between this approach and stochastic dominance rankings. This may be useful for improving the design of financial products.
- Many extensions: With Steven Vanduffel (Brussels),
  - Generalization in a multidimensional market (also with Mateusz Maj (Brussels)).
  - Derivation of upper and lower bounds for indifference prices of insurance claims.
  - Extensions with state-dependent constraints.

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Proof of Main Result

Assume that ξ_T is continuously distributed.
Consider a strategy with payoff X_T distributed as F. We define F^{-1} as follows:

$$F^{-1}(y) = \min\left\{x \ / \ F(x) \geq y\right\}.$$

The cost of the strategy with payoff X_T is

$$c(X_T) = E[\xi_T X_T].$$

Then,

$$E[\xi_T F_X^{-1}(1 - F_\xi(\xi_T))] \leqslant c(X_T) \leqslant E[\xi_T F_X^{-1}(F_\xi(\xi_T))]$$

It comes from the following property. Let $Z = F_Z^{-1}(U)$, then

$$E[F_Z^{-1}(U)\, F_X^{-1}(1 - U)] \leqslant E[F_Z^{-1}(U)\, X] \leqslant E[F_Z^{-1}(U)\, F_X^{-1}(U)]$$

$\Rightarrow$ **Bounds for financial claims.**

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