Path-dependent inefficient strategies and how to make them efficient.

Illustrated with the study

of a popular retail investment product

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Outline of the presentation

- What is cost-efficiency?
- Path-dependent payoffs are not cost-efficient.
- Consequences on the investors' preferences.
- Illustration with a popular investment product: the **locally-capped globally-floored** contracts (*highly path-dependent*).
- **Why do retail investors buy these contracts?**
  - Provide some explanations & evidence from the market.
    - Investors can overweight probabilities of getting high returns.
    - Locally-capped products are complex
  - Provide a simple model

Efficiency Cost

Dybvig (RFS 1988) explains how to compare two strategies by analyzing their respective efficiency cost.

It is a criteria independent of the agents' preferences.

What is the "efficiency cost"?

Efficiency Cost

- Given a strategy with payoff X_T at time T.
- Its no-arbitrage price P_X.
- F : X_T's distribution under the physical measure.

The distributional price is defined as:

$$PD(F) = \min_{\{Y_T \mid Y_T \sim F\}} \{\text{No-arbitrage Price of } Y_T\}$$

The "loss of efficiency" or "efficiency cost" is equal to:

$$P_X - PD(F)$$

Toy Example

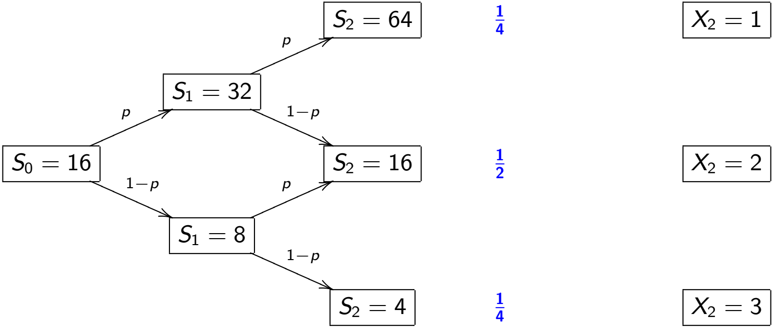
Consider :

- A market with 2 assets: a bond and a stock S.
- A discrete 2-period binomial model for the stock S.
- A financial contract with payoff X_T at the end of the two periods.
- An expected utility maximizer with utility U.

Let's illustrate what the "efficiency cost" is and why it is a criteria independent of agents' preferences.

Toy Example for X_2, a payoff at $T = 2$

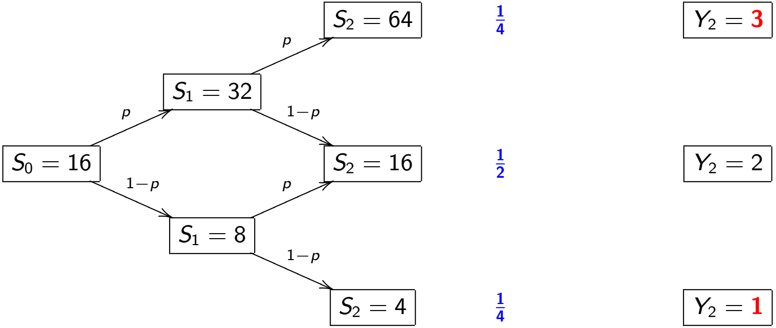
Real probabilities$=p = \frac{1}{2}$



$$E[U(X_2)] = \frac{U(1) + U(3)}{4} + \frac{U(2)}{2}$$

Y_2, a payoff at $T = 2$ distributed as X

Real probabilities$=p = \frac{1}{2}$

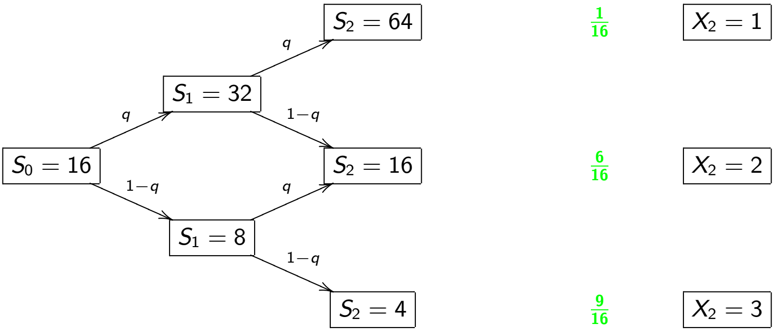


$$E[U(Y_2)] = \frac{U(\mathbf{3}) + U(\mathbf{1})}{4} + \frac{U(2)}{2}$$

(X and Y have the same distribution under the physical measure and thus the same utility)

X_2, a payoff at $T=2$

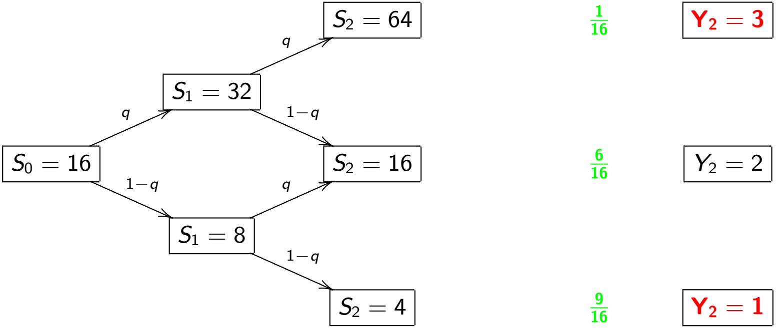
risk neutral probabilities$=q=\frac{1}{4}$.



$$P_X = \text{Price of } X = e^{-rT}\left(\frac{1}{16}+\frac{6}{16}2+\frac{9}{16}3\right)=\frac{5}{2}e^{-rT}$$

Y_2, a payoff at $T = 2$

risk neutral probabilities$=q = \frac{1}{4}$.

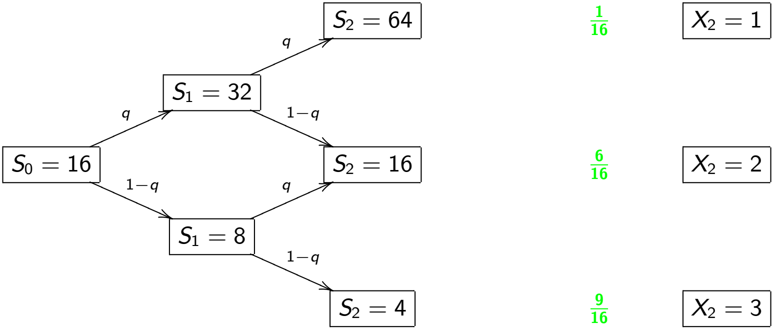


$$P_Y = e^{-rT}\left(\frac{1}{16}3 + \frac{6}{16}2 + \frac{9}{16}1\right) = \frac{3}{2}e^{-rT}$$

$$P_X \;=\; \text{Price of } X = e^{-rT}\left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2}e^{-rT} \quad ,$$

Toy Example for X_2, a payoff at $T = 2$

risk neutral probabilities$=q = \frac{1}{4}$.

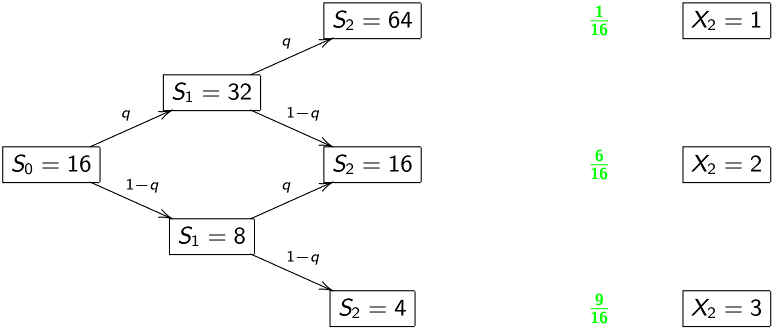


$$P_D = Cheapest = \frac{3}{2}e^{-rT}$$

$$P_X \ = \ \text{Price of } X = \frac{5}{2}e^{-rT} \quad ,$$

Toy Example for X_2, a payoff at $T = 2$

risk neutral probabilities$=q = \frac{1}{4}$.

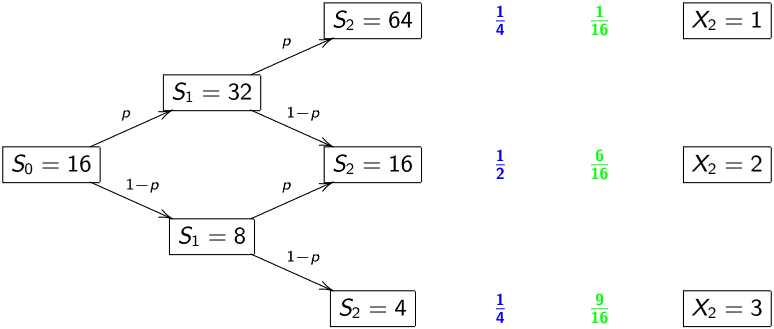


$$P_D = Cheapest = \frac{3}{2}e^{-rT}$$

$$P_X = \text{Price of } X = \frac{5}{2}e^{-rT} \quad , \quad \text{Efficiency cost} = P_X - P_D$$

Toy Example for X_2, a payoff at $T = 2$

Real probabilities$=p = \frac{1}{2}$ and **risk neutral** probabilities$=q = \frac{1}{4}$.



$$E[U(X_2)] = \frac{U(1)+U(3)}{4} + \frac{U(2)}{2} \quad , \quad P_D = Cheapest = \frac{3}{2}e^{-rT}$$

$$P_X \;=\; \text{Price of } X = \frac{5}{2}e^{-rT} \quad , \quad \text{Efficiency cost} = P_X - P_D$$

Cost-efficiency in a general arbitrage-free model

- In an **arbitrage-free market**, there exists at least one state price process $(\xi_t)_t$. We choose one to construct a pricing operator.
- The **cost of a strategy** (or of a financial investment contract) with terminal payoff X_T is given by:

$$c(X_T) = E[\xi_T X_T]$$

- The **"distributional price"** of a cdf F is defined as:

$$PD(F) = \min_{\{Y \mid Y \sim F\}} \{c(Y)\}$$

  where $\{Y \mid Y \sim F\}$ is the set of r.v. distributed as X_T is.
- The **efficiency cost** is equal to:

$$c(X_T) - P_D(F)$$

Minimum Cost-efficiency

Given a payoff X_T with cdf F. We define its inverse F^{-1} as follows:

$$F^{-1}(y) = \min\{x \;/\; F(x) \geq y\}.$$

Theorem

Define

$$X_T^* = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

then $X_T^* \sim F$ *and* X_T^* *is unique a.s. such that:*

$$PD(F) = c(X_T^*)$$

Path-dependent payoffs are inefficient

Corollary

In general, path-dependent derivatives are not cost-efficient. To be cost-efficient, the payoff of the derivative has to be of the following form:

$$X_T^* = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

Thus, it has to be a European derivative written on the state-price process at time T. It becomes a European derivative written on the stock S_T as soon as the state-price process ξ_T can be expressed as a function of S_T.

Monotonic Payoffs may be efficient

Corollary

Consider a derivative with a payoff X_T which could be written as:

$$X_T = h(\xi_T)$$

Then X_T is cost efficient if and only if h is non-increasing. Moreover, if X_T is cost-efficient, it satisfies:

$$X_T = X_T^* = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right) a.s.$$

Black and Scholes model (Dybvig (1988))

Any path-dependent financial derivative is inefficient. Indeed

$$\xi_T = a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = \exp\left(\frac{\theta}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)T - \left(r + \frac{\theta^2}{2}\right)T\right), b = \frac{\theta}{\sigma}, \theta = \frac{\mu - r}{\sigma}$.

To be cost-efficient, the payoff has to be written as:

$$X^* = F^{-1}\left(1 - F_{\xi}\left(a\left(\frac{S_T}{S_0}\right)^{-b}\right)\right)$$

It is a European derivative written on the stock S_T (**and the payoff is increasing with** S_T when $\mu > r$).

Lévy model with the Esscher transform (Vanduffel et al. (2008))

Any path-dependent financial derivative is inefficient. Indeed

$$\xi_t = e^{-rt}\frac{e^{h\frac{S_t}{S_0}}}{m_t(h)}$$

where $h \in \mathbb{R}$ is the unique real number such that $\xi_t S_t$ is a martingale under the physical measure.

$m_t(h)$ is a normalization factor such that $f_t^{(h)}$ defined by $f_t^{(h)}(x) = \frac{e^{hx} f_t(x)}{m_t(h)}$ is a density where f_t denotes the density of S_t under the physical measure.

To be cost-efficient, the payoff has to be written as:

$$X_T^* = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

It is a European derivative written on the stock S_T (**and the payoff is increasing with** S_T when $h < 0$).

The least efficient payoff

Theorem

Let F be a cdf such that $F(0) = 0$. Consider the following optimization problem:

$$\max_{\{Z \mid Z \sim F\}} \{c(Z)\}$$

The strategy Z_T^* that generates the same distribution as F with the highest cost can be described as follows:

$$Z_T^* = F^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

Consider a strategy with payoff X_T distributed as F. The cost of this strategy satisfies:

$$P_D(F) \leqslant c(X_T) \leqslant E[\xi_T F^{-1}(F_\xi(\xi_T))] = \int_0^1 F_\xi^{-1}(v) F^{-1}(v) dv$$

Put option in Black and Scholes model

Assume a strike K. Its payoff is given by:

$$L_T = (K - S_T)^+$$

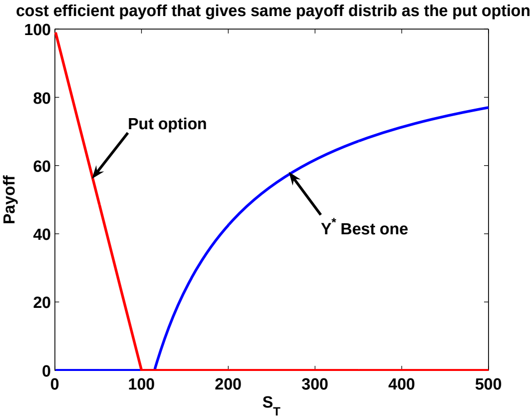
The payoff that has the **lowest** cost and is distributed such as the put option is given by:

$$Y_T^* = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

The payoff that has the **highest** cost and is distributed such as the put option is given by:

$$Z_T^* = F_L^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

Cost-efficient payoff of a Put



With $\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional Price of the put $= 3.14$

Price of the put $= 5.57$

Efficiency loss for the put $=$ 5.57-3.14$=$ 2.43

Up and Out Call option in Black and Scholes model

Assume a strike K and a barrier threshold $H > K$. Its payoff is given by:

$$L_T = (S_T - K)^+ \mathbb{1}_{\max_{0\leqslant t\leqslant T}\{S_t\}\leqslant H}$$

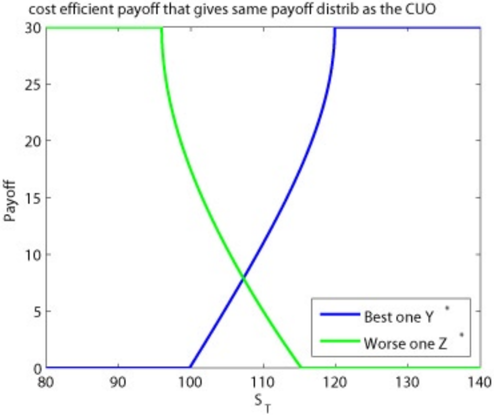
The payoff that has the **lowest** cost and is distributed such as the barrier up and out call option is given by:

$$Y_T^* = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

The payoff that has the **highest** cost and is distributed such as the barrier up and out call option is given by:

$$Z_T^* = F_L^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

Cost-efficient payoff of a Call up and out



With $\sigma = 20\%, \mu = 9\%, S_0 = 100,\ T = 1$ year, strike $K = 100,\ H = 130$

Distributional Price of the CUO = 9.7374

Price of CUO = P_{cuo}

Worse case = 13.8204

Efficiency loss for the CUO = P_{cuo}-9.7374

Utility independent criteria

Denote by

- X_T the final wealth of the investor,
- $V(X_T)$ the objective function of the agent,

Assumptions (adopted by Dybvig (JoB1988,RFS1988))

1. **Agents' preferences depend only on the probability distribution of terminal wealth:** "state-independent" preferences. (if $X_T \sim Z_T$ then: $V(X_T) = V(Z_T)$.)
2. **Agents prefer "more to less"**: if c is a non-negative random variable $V(X_T + c) \geqslant V(X_T)$.
3. The market is perfectly liquid, no taxes, no transaction costs, no trading constraints (in particular short-selling is allowed).
4. The market is **arbitrage-free**.

For any inefficient payoff, there exists another strategy that should be preferred by these agents.

Utility independent criteria

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Link with First Stochastic Dominance

Theorem

Consider a payoff X_T with cdf F,

1. *Taking into account the initial cost of the derivative, the cost-efficient payoff X_T^* of the payoff X_T dominates X_T in the first order stochastic dominance sense :*

$$X_T - c(X_T)e^{rT} \prec_{fsd} X_T^* - P_D(F)e^{rT}$$

2. *The dominance is strict unless X_T is a non-increasing function of ξ_T.*

Thus the result is true for any preferences that respect first stochastic dominance. This possibly includes state-dependent preferences.

How to explain the demand for inefficient payoffs (path-dependent, non-monotonic...)?

1. Needs may be state-dependent
   - **Presence of a background risk :**
     - Hedging a long position in the market index S_T (background risk) by purchasing a put option P_T.
     - the background risk can be path-dependent,
   - **Presence of a stochastic benchmark:** If the investor wants to outperform a given (stochastic) benchmark Γ such that:

     $$P\left\{\omega \in \Omega \,/\, W_T(\omega) > \Gamma(\omega)\right\} \geqslant \alpha$$

     Her preferences are now state-dependent preferences.
   - **Intermediary consumptions, additional constraints**
2. **Presence of another source of uncertainty.** The state-price process is not always a decreasing function of the asset price at maturity (non-markovian stochastic interest rates for instance)

What do popular contracts in the US look like?

Structured products sold by **banks** and Variable Annuities, Equity Indexed Annuities sold by **insurance companies** have become very popular. Structured product designs can be modified and extended in countless ways. Here are some of them:

- Guaranteed **floor**, Upper limits or **caps**
- Path-dependent payoffs (Asian, lookback, barrier)
- Multi-period based returns: **locally-capped contracts**

We concentrate our study on the latter ones. Biased beliefs may be an important reason to explain the demand among retail investors.

Example of a locally-capped contract

Quarterly Cap 6%

Quarter	Raw Index Return %	Capped return%
1	5	5
2	9	**6**
3	-10	**-10**
4	11	**6**

Payoff of a Quarterly Sum Cap = 5+6-10+6=7

Example of a locally-capped contract

- Issuer: JP Morgan Chase
- Underlying: S&P500
- Maturity: 5 years
- Initial investment: \$1,000
- Payoff$=\max\left(\$1,100\quad;\quad \$1,000+\textit{additional amount}\right)$
- In the prospectus dated June 22, 2004:
  "The additional amount will be calculated by the calculation agent by multiplying \$1,000 by the sum of the quarterly capped Index returns for each of the 20 quarterly valuation periods during the term of the notes."

Payoff of a locally-capped globally-floored contract

- Initial investment= \$1,000
- Minimum guaranteed rate $g = 10\%$ at maturity $T = 5$ years.
- Local Cap $c = 6\%$ on the quarterly return.

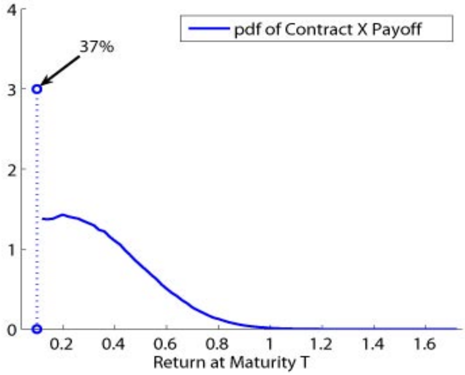
$$X_T = 1,000 + 1,000 \max\left(g \, , \, \sum_{i=1}^{20} \min\left(c, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right) \right)$$

- The contract consists of:
  - a zero coupon bond with maturity amount $\$1,100$.
  - a complex option component

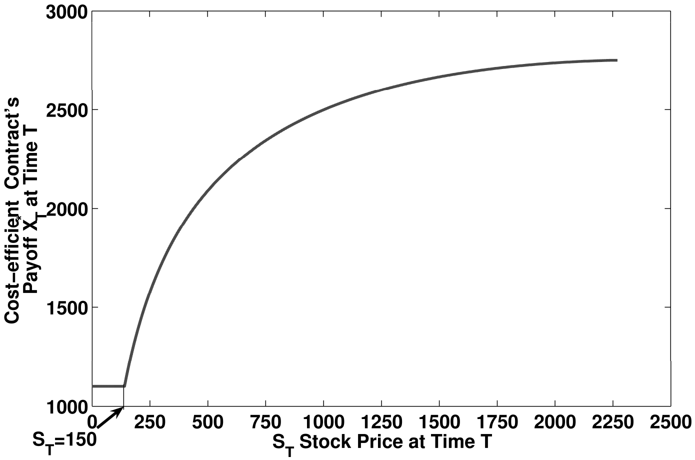
Distribution of the Payoff of a Quarterly Sum Cap

1. The distribution of the payoff of a Quarterly Sum Cap is extremely difficult for investors to have a realistic representation of the sum of periodically capped returns.
2. The reason stems from how the cap affects the final distribution of returns.

- Minimum guaranteed rate of 10% (global floor) over T years.
- Density of the payoff under the Quarterly Sum Cap (X).
- Parameters are set to $r = 5\%$, $\delta = 2\%$, $\mu = 0.09$, $\sigma = 15\%$.



LC contracts are not cost-efficient. Let F be the distribution of the payoff of a locally-capped. The payoff X^* should be preferred (lower cost & same utility), $S_0 = 100$, $T = 5$ years.



Summary

But then, why do retail investors buy locally-capped contracts?
They should choose simpler contracts that are not path-dependent.

- Investors are optimistic: investors may be influenced by the bias in the hypothetical projections displayed in the prospectuses to **overweight** the probabilities of receiving the maximum possible return.
- The **complexity** of the contract confuses investors and they make inappropriate choices (Carlin (2006)).

Structured Products Corp., the Depositor

25,300,000 TIERS® Principal-Protected Minimum Return Trust Certificates

(Interest on Final Scheduled Distribution Date
Based Upon the Nasdaq-100 Index®)

Due January 30, 2009

($10 Principal Amount Per Certificate)

issued by

TIERS® Principal-Protected Minimum Return Asset Backed Certificates Trust Series Nasdaq 2003-13

Ambac

Payments to the Trust Guaranteed Pursuant to the
Terms of a Financial Guaranty Insurance Policy

Characteristic of this locally-capped contract

- AMEX Ticker: NAS
- This product is based on the Nasdaq under the name *NAS: Nasdaq-100 Index TIERS*.
- The initial investment is \$10
- The maturity payoff is a compounded monthly-capped returns
- Capped at 5.5% per month.
- In the prospectus, there are 7 hypothetical examples.

***Example 1:** The value of the Nasdaq-100 Index as of the final scheduled distribution date is greater than its value at issuance and the Nasdaq-100 Index appreciated by 3.00% (an amount less than the periodic appreciation cap) during each period throughout the term of the certificates:*

	2003		2004		2005		2006		2007		2008		2009	
	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return
January			1,515	3.00%	2,160	3.00%	3,079	3.00%	4,390	3.00%	6,259	3.00%	8,924	3.00%
February			1,560	3.00%	2,224	3.00%	3,171	3.00%	4,522	3.00%	6,447	3.00%		
March			1,607	3.00%	2,291	3.00%	3,267	3.00%	4,657	3.00%	6,640	3.00%		
April..........			1,655	3.00%	2,360	3.00%	3,365	3.00%	4,797	3.00%	6,839	3.00%		
May			1,705	3.00%	2,431	3.00%	3,465	3.00%	4,941	3.00%	7,045	3.00%		
June			1,756	3.00%	2,504	3.00%	3,569	3.00%	5,089	3.00%	7,256	3.00%		
July			1,809	3.00%	2,579	3.00%	3,677	3.00%	5,242	3.00%	7,474	3.00%		
August	1,307	3.00%	1,863	3.00%	2,656	3.00%	3,787	3.00%	5,399	3.00%	7,698	3.00%		
September	1,346	3.00%	1,919	3.00%	2,736	3.00%	3,900	3.00%	5,561	3.00%	7,929	3.00%		
October	1,386	3.00%	1,976	3.00%	2,818	3.00%	4,017	3.00%	5,728	3.00%	8,167	3.00%		
November	1,428	3.00%	2,036	3.00%	2,902	3.00%	4,138	3.00%	5,900	3.00%	8,412	3.00%		
December	1,471	3.00%	2,097	3.00%	2,989	3.00%	4,262	3.00%	6,077	3.00%	8,664	3.00%		

$$
\begin{aligned}
\text{Index return} = [\,&(1.00 + 0.03) \times (1.00 + 0.03) \times (1.00 + 0.03) \times (1.00 + 0.03) \times (1.00 + 0.03) \times \\
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&(1.00 + 0.03) \times (1.00 + 0.03) \times (1.00 + 0.03) \times (1.00 + 0.03) \times (1.00 + 0.03) \times (1.00 + 0.03) \times \\
&(1.00 + 0.03)\,]\ \text{minus } 1.00 = 6.0349 \text{ or } 603.49\%.
\end{aligned}
$$

Interest distribution amount $= \$10.00 \times 6.0349 = \$60.35$

Payment on the final scheduled distribution date $= \$10.00 + \$60.35 = \$70.35$ per certificate.

Example 2: The value of the Nasdaq-100 Index as of the final scheduled distribution date is greater than its value at issuance and the Nasdaq-100 Index appreciated by 5.50% (an amount equal to the periodic appreciation cap) during each period throughout the term of the certificates:

	2003		2004		2005		2006		2007		2008		2009	
	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return
January			1,749	5.50%	3,325	5.50%	6,322	5.50%	12,020	5.50%	22,852	5.50%	43,447	5.50%
February			1,845	5.50%	3,508	5.50%	6,670	5.50%	12,681	5.50%	24,109	5.50%		
March			1,947	5.50%	3,701	5.50%	7,037	5.50%	13,378	5.50%	25,435	5.50%		
April.........			2,054	5.50%	3,905	5.50%	7,424	5.50%	14,114	5.50%	26,834	5.50%		
May			2,167	5.50%	4,120	5.50%	7,832	5.50%	14,891	5.50%	28,310	5.50%		
June			2,286	5.50%	4,346	5.50%	8,263	5.50%	15,710	5.50%	29,867	5.50%		
July			2,412	5.50%	4,585	5.50%	8,717	5.50%	16,574	5.50%	31,510	5.50%		
August	1,338	5.50%	2,544	5.50%	4,837	5.50%	9,197	5.50%	17,485	5.50%	33,243	5.50%		
September	1,412	5.50%	2,684	5.50%	5,103	5.50%	9,703	5.50%	18,447	5.50%	35,071	5.50%		
October	1,490	5.50%	2,832	5.50%	5,384	5.50%	10,236	5.50%	19,461	5.50%	37,000	5.50%		
November	1,571	5.50%	2,988	5.50%	5,680	5.50%	10,799	5.50%	20,532	5.50%	39,035	5.50%		
December	1,658	5.50%	3,152	5.50%	5,993	5.50%	11,393	5.50%	21,661	5.50%	41,182	5.50%		

Index return = [(1.00 + 0.055) × (1.00 + 0.055)] minus 1.00 = 33.2501 or 3,325.01%.

This is the maximum possible index return.

Interest distribution amount = \$10.00 × 33.2501 = \$332.50

Because the periodic capped return for any reset period will not in any circumstances be greater than 5.50%, \$332.50 is the maximum possible interest distribution amount.

Example 3: The value of the Nasdaq-100 Index as of the final scheduled distribution date is greater than its value at issuance and the Nasdaq-100 Index appreciated by 7.00% (an amount greater than the periodic appreciation cap) during each period throughout the term of the certificates:

	2003		2004		2005		2006		2007		2008		2009	
	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return
January			1,904	5.50%	4,288	5.50%	9,656	5.50%	21,748	5.50%	48,980	5.50%	110,313	5.50%
February ...			2,037	5.50%	4,588	5.50%	10,332	5.50%	23,270	5.50%	52,409	5.50%		
March			2,180	5.50%	4,909	5.50%	11,055	5.50%	24,899	5.50%	56,078	5.50%		
April			2,332	5.50%	5,252	5.50%	11,829	5.50%	26,642	5.50%	60,003	5.50%		
May			2,495	5.50%	5,620	5.50%	12,657	5.50%	28,507	5.50%	64,203	5.50%		
June			2,670	5.50%	6,013	5.50%	13,543	5.50%	30,502	5.50%	68,697	5.50%		
July			2,857	5.50%	6,434	5.50%	14,491	5.50%	32,638	5.50%	73,506	5.50%		
August	1,357	5.50%	3,057	5.50%	6,885	5.50%	15,506	5.50%	34,922	5.50%	78,652	5.50%		
September ..	1,452	5.50%	3,271	5.50%	7,367	5.50%	16,591	5.50%	37,367	5.50%	84,157	5.50%		
October	1,554	5.50%	3,500	5.50%	7,882	5.50%	17,753	5.50%	39,983	5.50%	90,048	5.50%		
November ..	1,663	5.50%	3,745	5.50%	8,434	5.50%	18,995	5.50%	42,781	5.50%	96,352	5.50%		
December ..	1,779	5.50%	4,007	5.50%	9,025	5.50%	20,325	5.50%	45,776	5.50%	103,096	5.50%		

*Actual return on the Nasdaq-100 Index during each reset period is 7.00%, but because of the 5.50% cap the periodic capped return would be 5.50%.

Index return = [(1.00 + 0.055) × (1.00 + 0.055) × (1.00 + 0.055) (1.00 + 0.055) × (1.00 + 0.055)] minus 1.00 = 33.2501 or 3,325.01%.

This is the maximum possible index return.

Interest distribution amount = \$10.00 × 33.2501 = \$332.50

Because the periodic capped return for any reset period will not in any circumstances be greater than 5.50%, \$332.50 is the maximum possible interest distribution amount.

Payment on the final scheduled distribution date = \$10.00 + \$332.50 = \$342.50 per certificate.

This is the maximum possible payment on the final scheduled distribution date.

Example 4: The value of the Nasdaq-100 Index as of the final scheduled distribution date is less than its value at issuance and the Nasdaq-100 Index declined steadily throughout the term of the certificates:

	2003		2004		2005		2006		2007		2008		2009	
	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return
January			1,082	−1.30%	971	−1.90%	741	−2.50%	525	−3.10%	346	−3.70%	211	−4.30%
February ...			1,166	−1.35%	952	−1.95%	722	−2.55%	509	−3.15%	333	−3.75%		
March			1,149	−1.40%	933	−2.00%	703	−2.60%	493	−3.20%	320	−3.80%		
April			1,133	−1.45%	914	−2.05%	685	−2.65%	477	−3.25%	308	−3.85%		
May			1,116	−1.50%	894	−2.10%	666	−2.70%	461	−3.30%	296	−3.90%		
June			1,098	−1.55%	875	−2.15%	648	−2.75%	445	−3.35%	284	−3.95%		
July			1,081	−1.60%	856	−2.20%	630	−2.80%	430	−3.40%	273	−4.00%		
August	1,255	−1.05%	1,063	−1.65%	837	−2.25%	612	−2.85%	415	−3.45%	262	−4.05%		
September ..	1,241	−1.10%	1,045	−1.70%	817	−2.30%	594	−2.90%	401	−3.50%	251	−4.10%		
October	1,227	−1.15%	1,027	−1.75%	798	−2.35%	577	−2.95%	387	−3.55%	241	−4.15%		
November ..	1,212	−1.20%	1,008	−1.80%	779	−2.40%	559	−3.00%	373	−3.60%	231	−4.20%		
December ..	1,197	−1.25%	990	−1.85%	760	−2.45%	542	−3.05%	359	−3.65%	221	−4.25%		

Index return $= [\,(1.00 + -0.0105) \times (1.00 + -0.0110) \times (1.00 + -0.0115) \times (1.00 + -0.0120) \times (1.00 + -0.0125) \times (1.00 + -0.0130) \times (1.00 + -0.0135) \times (1.00 + -0.0140) \times (1.00 + -0.0145) \times (1.00 + -0.0150) \times (1.00 + -0.0155) \times (1.00 + -0.0160) \times (1.00 + -0.0165) \times (1.00 + -0.0170) \times (1.00 + -0.0175) \times (1.00 + -0.0180) \times (1.00 + -0.0185) \times (1.00 + -0.0190) \times (1.00 + -0.0195) \times (1.00 + -0.0200) \times (1.00 + -0.0205) \times (1.00 + -0.0210) \times (1.00 + -0.0215) \times (1.00 + -0.0220) \times (1.00 + -0.0225) \times (1.00 + -0.0230) \times (1.00 + -0.0235) \times (1.00 + -0.0240) \times (1.00 + -0.0245) \times (1.00 + -0.0250) \times (1.00 + -0.0255) \times (1.00 + -0.0260) \times (1.00 + -0.0265) \times (1.00 + -0.0270) \times (1.00 + -0.0275) \times (1.00 + -0.0280) \times (1.00 + -0.0285) \times (1.00 + -0.0290) \times (1.00 + -0.0295) \times (1.00 + -0.0300) \times (1.00 + -0.0305) \times (1.00 + -0.0310) \times (1.00 + -0.0315) \times (1.00 + -0.0320) \times (1.00 + -0.0325) \times (1.00 + -0.0330) \times (1.00 + -0.0335) \times (1.00 + -0.0340) \times (1.00 + -0.0345) \times (1.00 + -0.0350) \times (1.00 + -0.0355) \times (1.00 + -0.0360) \times (1.00 + -0.0365) \times (1.00 + -0.0370) \times (1.00 + -0.0375) \times (1.00 + -0.0380) \times (1.00 + -0.0385) \times (1.00 + -0.0390) \times (1.00 + -0.0395) \times (1.00 + -0.0400) \times (1.00 + -0.0405) \times (1.00 + -0.0410) \times (1.00 + -0.0415) \times (1.00 + -0.0420) \times (1.00 + -0.0425) \times (1.00 + -0.0430)\,]$ minus $1.00 = -0.8335$ or -83.35% but the index return cannot be less than 7.00%

Interest distribution amount = \$10.00 × 0.07 = \$0.70

Payment on the final scheduled distribution date = \$10.00 + \$0.70 = \$10.70 per certificate, the amount of your original investment plus the minimum return of 7.00%.

Example 5: ***The value of the Nasdaq-100 Index as of the final scheduled distribution date is greater than its value at issuance and the Nasdaq-100 Index increased steadily throughout all but one of the reset periods during the term of the certificates. If the decline is greater than or equal to approximately 96.71% for one reset period, the index return will not be greater than the minimum return of 7.00%.***

	2003		2004		2005		2006		2007		2008		2009	
	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return	Index Level	Capped Return
January			1,749	5.50%	3,325	5.50%	6,322	5.50%	12,020	5.50%	713	5.50%	1,355	5.50%
February			1,845	5.50%	3,508	5.50%	6,670	5.50%	12,681	5.50%	752	5.50%		
March			1,947	5.50%	3,701	5.50%	7,037	5.50%	13,378	5.50%	793	5.50%		
April			2,054	5.50%	3,905	5.50%	7,424	5.50%	14,114	5.50%	837	5.50%		
May			2,167	5.50%	4,120	5.50%	7,832	5.50%	14,891	5.50%	883	5.50%		
June			2,286	5.50%	4,346	5.50%	8,263	5.50%	15,710	5.50%	931	5.50%		
July			2,412	5.50%	4,585	5.50%	8,717	5.50%	517	-96.71%	983	5.50%		
August	1,338	5.50%	2,544	5.50%	4,837	5.50%	9,197	5.50%	545	5.50%	1,037	5.50%		
September	1,412	5.50%	2,684	5.50%	5,103	5.50%	9,703	5.50%	575	5.50%	1,094	5.50%		
October	1,490	5.50%	2,832	5.50%	5,384	5.50%	10,236	5.50%	607	5.50%	1,154	5.50%		
November	1,571	5.50%	2,988	5.50%	5,680	5.50%	10,799	5.50%	640	5.50%	1,217	5.50%		
December	1,658	5.50%	3,152	5.50%	5,993	5.50%	11,393	5.50%	675	5.50%	1,284	5.50%		

$$
\begin{gathered}
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0.055)]\text{ minus } 1.00 = 0.0681 \text{ or } 6.81\% \text{ but the index return cannot be less than } 7.00\%
\end{gathered}
$$

Interest distribution amount $= \$10.00 \times 0.07 = \$0.70$

Payment on the final scheduled distribution date $= \$10.00 + \$0.70 = \$10.70$ per certificate, the amount of your original investment plus the minimum return of 7.00% (even though the value of the Nasdaq-100 Index increased in all but one of the reset periods).

Observations

- Most outrageous set of unrealistic assumptions we observed.
- In the 3 first examples, the final payoffs are respectively $1.03^{66} = \$60.35$, $1.055^{66} = \$332.5$, $1.055^{66} = \$332.5$.
- Empirical probability of a monthly return exceeding 5.5% is 0.2 (1971-2008).
- Assuming an i.i.d. distribution of the monthly returns, the probability of the maximum possible return is

$$0.2^{66} = 7 \times 10^{-47}$$

  which is an impossible event.
- Getting returns such as in Examples 4 and 5 have an historical probability of about 50% of taking place.
- Maximum value of the compounded return of 66 consecutive monthly-capped returns is 2.7 (end in May 1996).
- These securities are also subject to default risk.

Overview

- Our analysis of the hypothetical examples presented in the 39 prospectuses (39 locally-capped globally-floored contracts out of 208 index-linked notes as of October 2006 listed on AMEX) reveals that the above description is common practice.
- All issuers provide in their prospectus 4 to 7 hypothetical examples. One or two of the first three examples assumes that the investor receives the maximum possible return.
- We suggest that including these illustrations as hypothetical scenarios provides very concrete evidence of attempts to overweight the probabilities of obtaining the maximum possible return.

Local Cap vs Global Cap

- Initial investment= \$1,000
- Maturity $T = 5$ years
- Let $g = 10\%$ be the minimum guaranteed rate.
- Y_T: Globally-capped (with global Cap C)

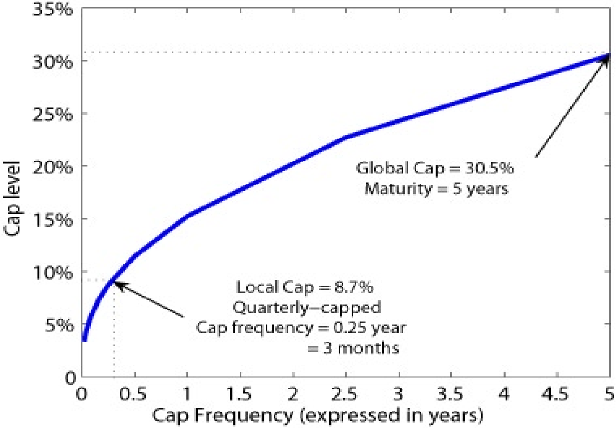
$$Y_T = 1,000 + 1,000 \max\left(g \ , \ \min\left(C, \frac{S_T - S_0}{S_0} \right) \right)$$

(long position in a bond and in a standard call option and short position in another standard call option.)

- X_T: Locally-Capped (Local Cap c on the quarterly return).

$$X_T = 1,000 + 1,000 \max\left(g \ , \ \sum_{i=1}^{20} \min\left(c, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right) \right)$$

How to perform the comparison?



Parameter values are $r = 5\%$, $\delta = 2\%$, $\sigma = 15\%$.
Same no-arbitrage prices along the curve.

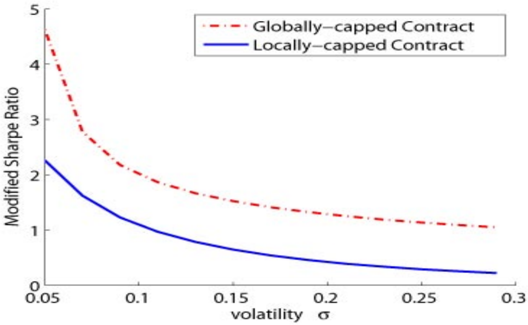
Mean Variance Investors

- Let Z_0 be the initial investment
- Let the guarantee be $(1+g)Z_0$ at the maturity T.
- We define the **modified Sharpe ratio** as follows

$$R_Z = \frac{\mathrm{E}[Z_T] - Z_0(1+g)}{\mathrm{std}(Z_T)}$$

- We compute this ratio for the quarterly-capped contract R_X and for the globally-capped contract R_Y.

Mean Variance Investors



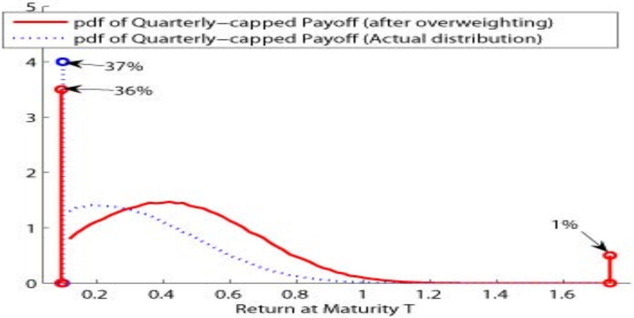
- The Quarterly Sum cap has a quarterly cap of 8.7%, a global floor $g = 10\%$ and a maturity $T = 5$ years.
- For each volatility, the global cap is such that the GC contract has the same no-arbitrage price as the 8.7% quarterly-capped (which is equal to 920$).
- Other parameters $r = 5\%$, $\delta = 2\%$, $\mu = 0.09$.

Overweighting Technique

1. increase the drift of the underlying index
2. add a lump of probability at the right end of the distribution.

Density of the payoff under the Quarterly Sum Cap (X) with an additional expected annual Index return of 5%.

The quarterly cap is $c = 8.7\%$, $r = 5\%$, $\mu = 9\%$, $\delta = 2\%$, $\sigma = 15\%$.



Impact on Decision Making

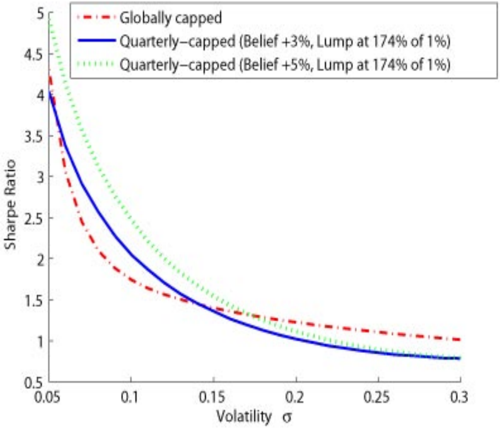
- Modified Sharpe ratio using the new measure for the quarterly Sum Cap and the original measure for the other contract:

$$\tilde{R}_X = \frac{\mathrm{E}_Q[Z_T] - Z_0(1+g)}{\mathrm{std}_Q(Z_T)}$$

- Compare of $\tilde{R}_X$ with R_Y
- 8.7% quarterly cap, $g = 10\%$, $T = 5$ years.
- Other parameters $r = 5\%$, $\delta = 2\%$, $\mu = 0.09$.

Impact on Decision Making

The quarterly-capped contract has a 8.7% quarterly cap, $g = 10\%$, $T = 5$ years. For each volatility, the cap of the globally-capped contract is such that the contract has the same no-arbitrage price as the 8.7% quarterly-capped contract. Investors overweight the tail of the distributions. Other parameters $r = 5\%$, $\delta = 2\%$, $\mu = 0.09$.



Conclusions of this study

- We describe some popular designs in the market: locally-capped contracts.
- The demand for these complex products is puzzling.
- We provide a possible explanation based on investor misperception of the return distribution where low probability events of high returns are overweighted.
- We provide evidence that this tendency is encouraged by the hypothetical examples in the prospectus supplements.

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